A Barrel of Graphs
Key Stage 4 Revision Programme

Pupil Name

Handbook Designed by Dr Stephen Flood
Guidance and Contents for this booklet

This is by far the longest booklet we have shared so far, so we are sharing some guidance and a contents page.

These courses are at their best when completed in a timely fashion and using all the assessments attached. If necessary, individual lessons can be removed and used in isolation but it is not recommended.

We recommend you begin with the baseline assignment and mark it yourself using the mark scheme. This gives you an indication of your starting point – what your strengths and areas for development in straight line graphs may be. Then do each lesson in order and finish with the final assignment. You can then compare your baseline and final assignment to see how much you’ve learned and if you still have some areas for development.

This booklet contains:
1. Four self-contained lessons on straight line graphs
2. An answer booklet
3. A baseline assignment
4. A final assignment
5. Baseline assignment mark scheme
6. Final assignment mark scheme
## Timetable and Assignment Submission

### Timetable – Tutorials

<table>
<thead>
<tr>
<th>Tutorial</th>
<th>Date</th>
<th>Time</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Baseline assessment)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 (Final assessment)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 (Feedback)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Course Rationale: Why this Book on Graphs? 4
Glossary of Keywords 5
Tutorial 1: Preliminaries 7
Tutorial 2: Straight line graphs 8
A Understanding \( y = mx + c \) 8
B Parallel and Perpendicular Lines 14
C Midpoints and Ratios on Straight Lines 17
Tutorial 3: Plotting and Interpreting Straight Line Graphs 22
A Plotting from \( y = mx + c \) 22
B Finding the Equation of a Line Given Points or Gradient 29
C Solving Simultaneous Equations with Graphs 32
Tutorial 4: Quadratic Graphs 37
A Plotting a Quadratic Graph from an Equation 37
B Interpreting Quadratic Graphs to Find Roots and Values 45
C Approximating Minima and Lines of Symmetry Using Graphs 48
Tutorial 5: Advanced Graphs 52
A Solving Quadratic Simultaneous Equations Using a Graph 52
B Recognising Advanced Graphs 60
C Plotting Advanced Graphs 65
Tutorial 6: Revision and Final Assessment 69
Tutorial 7: Feedback 70
Course Rationale

When revising maths, everyone has their own methods. One that is very rarely chosen is simply PRACTICE! It’s obvious that any musician will spend far more time practicing their instrument than reading and highlighting – the same is true of maths!

Naturally, when faced with topics we’re good at and those we’re not so confident with we tend to favour the good topics and avoid the difficult ones. This book is designed specially to give a helping hand with one topic often found challenging and done poorly by GCSE students across the country.

By offering key examples, varied and extended practice examples you’ll gain confidence and the key skills necessary to boost your knowledge of GCSE maths and get those top grades in year 11.
<table>
<thead>
<tr>
<th>Word</th>
<th>Definition</th>
<th>In a sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Midpoint</td>
<td>A point in the middle of a line segment or pair of points</td>
<td>The midpoint of (1,1) and (5,9) is (3,4)</td>
</tr>
<tr>
<td>Plot</td>
<td>To make a graph by marking a number of points on a grid</td>
<td>I would like you to plot the graph $y = 3x + 2$ on the grid</td>
</tr>
<tr>
<td>Gradient</td>
<td>The measure of how steep a curve is</td>
<td>The line $y = 2x$ has a smaller gradient than $y = 3x$ so it is less steep.</td>
</tr>
<tr>
<td>Intercept</td>
<td>The point on a graph where the line crosses an axis</td>
<td>The $y$-intercept of $y = 2x + 3$ is 3 because it crosses the $y$-axis at 3</td>
</tr>
<tr>
<td>Parallel lines</td>
<td>Straight lines which never cross. These lines have equal gradients.</td>
<td>The left and right edges of this book are parallel lines</td>
</tr>
<tr>
<td>Perpendicular lines</td>
<td>Straight lines which cross at a right angle to each other.</td>
<td>The $x$ and $y$ axes are perpendicular to each other because the angle between them is 90°</td>
</tr>
<tr>
<td>Quadrant</td>
<td>A quarter of a graph used to describe the regions where</td>
<td>The point with coordinates (-3, 4) is in the second quadrant</td>
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<tr>
<td></td>
<td>$\cdot$ $x$ and $y$ are both positive (first quadrant)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\cdot$ $x$ is negative, $y$ is positive (second quadrant)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\cdot$ both $x$ and $y$ are negative (third quadrant)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\cdot$ $x$ is positive and $y$ is negative (fourth quadrant)</td>
<td></td>
</tr>
<tr>
<td>Linear</td>
<td>A graph that is a straight line.</td>
<td>$y = 4x + 2$ is a linear graph</td>
</tr>
<tr>
<td>Axis (plural: axes)</td>
<td>Perpendicular lines used as references for plotting or interpreting points or graphs</td>
<td>The coordinates (0,3) and (0,3) both lie on the $y$-axis</td>
</tr>
<tr>
<td>Simultaneous</td>
<td>At the same time: used in maths when there are two equations to be solved with the same unknowns</td>
<td>The equations $x + y = 1$ and $x - y = 3$ are simultaneously true. What are $x$ and $y$?</td>
</tr>
<tr>
<td>Quadratic</td>
<td>Refers to a quantity/equation/graph with highest power being a squared term</td>
<td>The graph $y = x^2 + 3$ is quadratic where $y = x + 3$ is not</td>
</tr>
<tr>
<td><strong>Parabola</strong> (plural: parabolae)</td>
<td>A type of curve created by plotting a quadratic equation</td>
<td>The graphs $y = x^2$ and $y = -x^2$ both produce a <strong>parabola</strong>, though they are reflections of one another</td>
</tr>
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</tr>
<tr>
<td><strong>Root</strong></td>
<td>The solution to an equation, typically seen with quadratic or cubic equations by setting equal to zero</td>
<td>The <strong>roots</strong> of $x^2 + 3x + 2$ are $x = -1$ and $x = -2$</td>
</tr>
<tr>
<td><strong>Minimum</strong> (plural: minima)</td>
<td>The smallest allowed value</td>
<td>The <strong>minimum</strong> value of $x^2$ is 0 since negative times negative is positive</td>
</tr>
<tr>
<td><strong>Turning Point/ Vertex</strong></td>
<td>A point on a curve where the gradient changes from positive to negative. With quadratics this is either a minimum or maximum and is also called the vertex</td>
<td>The <strong>turning point</strong> of the graph $y = (x - 1)^2$ is at (1,0)</td>
</tr>
<tr>
<td><strong>Cubic</strong></td>
<td>Refers to a quantity/equation/graph with highest power being a cubic term</td>
<td>The graph $y = x^3 + 3$ is <strong>cubic</strong> where $y = x^2 + 3$ is not</td>
</tr>
<tr>
<td><strong>Reciprocal</strong></td>
<td>The reciprocal of a number is found by dividing 1 by the number</td>
<td>The <strong>reciprocal</strong> of 5 is $\frac{1}{5}$</td>
</tr>
<tr>
<td><strong>Exponential</strong></td>
<td>Refers to a quantity/equation/graph with a term with $x$ as an exponent (power)</td>
<td>The graph $y = 2^x$ is <strong>exponential</strong>; it grows very quickly!</td>
</tr>
</tbody>
</table>
Baseline Assignment

At this stage you should take the baseline assignment. This helps you track your learning over this course. The mark scheme and assignment and available at the end of this document.
Lesson 1: Straight Line Graphs

Objectives this Lesson
A) Understanding \( y = mx + c \)
B) Being able to find midpoints and ratios on lines
C) Being able to find and recognise parallel and perpendicular lines

Starter
Try to think how you might solve the following problems:
- Find 3 possible pairs of coordinates so that \((-3,7)\) is the midpoint
  1)  
  2)  
  3)  
- Write down the equations of 3 lines parallel to \(3y + x = 7\)
  1)  
  2)  
  3)  

Objective A: Understanding \( y = mx + c \)

Every linear graph (every graph of a straight line) can be written in the form

\[ y = mx + c \]

where \(x\) and \(y\) are variables; quantities that vary depending on one another, such as ‘height and weight’ of a person or ‘age and cost’ of a phone. These appear on the axes of the graph. The other quantities \(m\) and \(c\) are called the gradient and \(y\)-intercept respectively. The \(y\)-intercept is the easiest to explain: it’s just the point where the straight line crosses. The gradient gives information about how steep the line is.

But what does the straight line graph actually mean? \( y = mx + c \) tells us that if we take a point \(x\), multiply it by \(m\) and add \(c\), we get \(y\). For example, \(y = 2x + 1\) with \(x = 1\) yields \(y = 3\). This tells us that \((1,3)\) is a solution of the equation. But also if \(x = 0\) then \(y = 1\) so \((0,1)\) is also a solution to the equation. It turns out that there are a lot of solutions to equations like this, and they lie on a straight line! The line we draw marks out every possible solution of the equation. Points that don’t lie on the line just denote \(x, y\) pairs that don’t ‘match up’ in the given equation.

The gradient of a straight line is defined by the formula

\[
\text{gradient} = \frac{\text{change in } y}{\text{change in } x}
\]

Or if the graph is know to go through coordinates \((x_1, y_1)\) and \((x_2, y_2)\)

\[
\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}
\]
In this formula, note it doesn’t matter which coordinate pair we call the \( x_1 \) and which we call the \( x_2 \) – we get the same result either way (try it!).

Useful facts to remember about the gradient:
- A positive gradient slopes upward (from bottom left to top right: see the green line)
- A negative gradient slopes downward (from top left to bottom right: see the red line)
- The size of the gradient tells us how steep the slope is: the bigger, the steeper.

**Worked Examples A**

**Worked Example 1**

Write down the gradients and intercepts of the following lines
(a) \( y = 12 - 12x \)
(b) \( x + 2y = 2 \)

For this question we need to remember that when we rearrange our line to be in the form \( y = mx + c \) then we can read off the gradient and intercept easily!

For the first part, we can write the equation as \( y = -12x + 12 \) (make sure to take care with the signs!) and then we can read off the gradient and intercept:

\[
\begin{align*}
\text{gradient} & = -12 \\
\text{intercept} & = 12
\end{align*}
\]

For the second line, a little more work is needed: first to isolate the \( y \) term by subtracting \( x \) from both sides \( x + 2y = 2 \)

\[
2y = 2 - x
\]

Then to get into the right form we must divide both sides by 2 (be careful to divide each term on both sides!)

\[
\begin{align*}
y & = 1 - \frac{x}{2}
\end{align*}
\]

Which we can again rearrange to make it clearer to read off the gradient and intercept

\[
\begin{align*}
\text{gradient} & = -\frac{1}{2} \\
\text{intercept} & = 1
\end{align*}
\]

**Worked Example 2**

(a) Work out the gradient of the line AB
\( P \) is the point \((6,-2)\).
(b) Work out the gradient of the lines AP and PB

Diagram NOT accurately drawn

\( A(-1, 2) \)

\( B(7, 5) \)
In this question we’ve been given two coordinates already. For part (a) we can simply read the difference in the \( y \)-coordinates by subtracting the \( y \)-coordinate of \( A \) from the \( y \)-coordinate of \( B \) and the same for the \( x \)-coordinates:

\[
m = \frac{5 - 2}{7 - (-1)} = \frac{3}{8}
\]

And we leave it positive since the graph is from bottom left to top right.

For the second part, it helps to mark on the question approximately where the point is. We can see from the coordinates which quadrant (quarter) of the graph it must be in, and we know it must be to the left of the point \( B \) (since it has a smaller \( x \)-coordinate)

Then we repeat the procedure above:

For the line \( AP \)

\[
m = \frac{2 - (-2)}{-1 - 6} = \frac{4}{-5}
\]

And as the line must be drawn from upper-left to lower-right the negative sign is correct.

Note: if the numbers have been entered into the formula correctly, we can circle them. If done incorrectly, they won’t be in a column.

\[
m = \frac{2 - 2}{-1 - 6} = \frac{0}{-7} = 0
\]

For the line \( PB \):

\[
m = \frac{5 - 2}{7 - 6} = \frac{3}{1} = 3
\]

And as the line must be drawn from lower-left to upper-right the no negative sign is needed!

---

**Worked Example 3**

Find

(a) the gradient
(b) the \( y \)-intercept and
(c) the equation
of the line in the figure

---

**Step 1: pick two exact points**

In order to get an accurate value for the gradient, two points on the graph are needed. We can make life much easier for ourselves by choosing the two points wisely. Here we can notice that our line is on a grid-line when \( x = 1 \) (\( y = 10 \)) and when \( x = 6 \) (\( y = 50 \)) so we can choose those as our two points, rather than struggle with decimals and guessing between grid-lines.
Step 2: Find the gradient
There are two methods for finding the gradient

**Method 1**
The easiest way to implement this is to draw the right-angled triangle between the two points to find the change in \( x \) and the change in \( y \).

Then we need to look at the direction of the slope for the sign of the gradient: it’s directed from bottom left to top right, so must be positive.

**Method 2**
Use the formula: let’s choose \((1,10)\) for the \((x_1,y_1)\) and \((6,50)\) for the \((x_2,y_2)\) (again note we could have chosen the other way around, it doesn’t matter! It only matters that we don’t mix up and use one of our points for \(x_1\) and the other point for \(y_1\))

\[
\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{50 - 10}{6 - 1} = \frac{40}{5} = 8
\]

Note: a gradient of 8 is a steep line (for every 1 step in \( x \) the line goes 8 in \( y \))! The reason it doesn’t look so steep on the graph is because the scale of the axes are different for \( x \) and \( y \! \). 

Step 3: find the \( y \)-intercept
Simply read the value from the \( y \)-axis where it touches the green line (and be careful with the scale – in this instance the major grid-line on the \( y \)-axis is 10, so a minor one is 2. (This isn’t always true on all graph scales, count to see how many minor grid-lines are in a major one!)

Step 4: write the equation of the line

We’ve already found the gradient and the intercept, so all that’s left is to write \( y = mx + c \) with our gradient \( m \) and our intercept \( c \):

\[
y = 8x + 2
\]

**Practice Questions A**

1) Find the gradient of the graph given by the green line shown
2) Find the gradient of the graph given by the blue line shown

3) Write down the gradient of the straight line with equation $3y = 2(3 - 2x)$

4) Write down the gradient of the straight line with equation $2 = 2y + 3x$
Challenge Question A

A scientist adjusts the current while measuring the voltage across an electronic device.

(a) The scientist plots a best-fit line. Find the gradient and y-intercept of this line.

(b) Given that Voltage ($V$) is related to the device’s resistance ($R$) and current ($I$) flowing through the circuit by the formula $V = IR$, find the resistance of the device.

Assessment Question A

What is the equation of the green line?

A) $y = -2x + 2.5$

B) $y = 2x + 2.5$

C) $y = \frac{1}{2}x + 2.5$

D) $y = -\frac{1}{2}x + 2.5$
Objective B: Parallel and Perpendicular Lines

As mentioned in the previous section, the gradient of a line indicates how steep the line is. Since parallel lines can never meet, they have the same steepness by definition! So

(1) Saying that two lines are parallel is the same as saying they have equal gradient

Perpendicular lines cross at right-angles: clearly if one has a positive gradient, the other must have a negative gradient (otherwise they couldn’t possibly cross at right angles!) but it turns out that

(2) Two lines are perpendicular if their gradients are negative reciprocals of one another
(You can prove this using Pythagoras – try it!)

In equations, if we have two lines \( y = m_1x + c_1 \) and \( y = m_2x + c_2 \) then

(1) The two lines are parallel if \( m_1 = m_2 \)
(2) The two lines are perpendicular if \( m_1 = \frac{-1}{m_2} \)
(Note that the +c terms don’t matter at all! Think: two lines can be parallel but next to each other or miles apart! Similarly, two lines can be perpendicular near the origin or really far away! That’s all the +c term is doing in this context)

Worked Examples B

Worked Example 1

Give an example of a line that is parallel to the line given by equation

\[
y = 3x + 4
\]

Since we need a line parallel to \( y = 3x + 4 \), we need to identify the gradient, which is the coefficient of \( x \), since the equation is already in the form \( y = mx + c \). The line’s gradient is 3, so any line of the form \( y = 3x + c \) will work. We choose a different \( c \), or else we’ll end up with the same line, so we choose

\[
y = 3x + 1
\]

Worked Example 2

Give an example of a line that is perpendicular to the line given by equation

\[
2y = -6x + 3
\]

Since we need a line perpendicular to \( 2y = -6x + 3 \), we need to identify the gradient, which requires an extra step this time: we need to get the equation in the form \( y = mx + c \) by dividing both sides by 2.

\[
y = -3x + \frac{3}{2}
\]

so we see the line in the question has gradient -3.

The gradient of the perpendicular line is given by the negative reciprocal of the original gradient:

\[
m = -\frac{1}{-3} = \frac{1}{3}
\]

So any line of the form \( y = \frac{1}{3}x + c \) will work. We choose a different \( c \), or else we’ll end up with the same line, so we choose \( y = \frac{1}{3}x + 2 \).
This question is more complex, but works in the same way as the previous example to start with: we see the word perpendicular, so we need to identify the gradient of the question’s line to find our gradient by the negative reciprocal (be sure to keep track of the minus signs and take care dividing by fractions!):

\[ m = -\frac{1}{-\frac{1}{2}} = -2 = 2 \]

So as before, we need a line of the form \( y = 2x + c \) but this time we don’t have free choice of the \( +c \) (if we choose \( c = 1 \) for example, we get \( y = 2x + 1 \), but we need the line to pass through \((1,10)\). Substituting this into \( y = 2x + 1 \) however leads to \( 10 = 2 \times 1 + 1 = 3 \), so \((1,10)\) isn’t on the line!)

To find \( c \), we use the fact that the line passes through \((1,10)\). This means substituting \( x = 1 \) and \( y = 10 \) into our line must work, so we can use that to find \( c \):

\[
\begin{align*}
y &= 2x + c \\
10 &= 2 \times 1 + c \\
8 &= c
\end{align*}
\]

So the line perpendicular to \( y = -1/2 + 10 \) that also passes through \((1,10)\) is \( y = 2x + 8 \)

**Practice Questions B**

1) For each of the below, state whether the lines are parallel, perpendicular or neither

   (a) \( y = -\frac{4}{9}x + 14 \) and \( y = -\frac{4}{9}x - 3 \)

   (b) \( y = -\frac{3}{2}x - 1 \) and \( y = \frac{3}{2}x + 3 \)

   (c) \( y = x - 6 \) and \( y = -x + 4 \)

   (d) \( y = \frac{3}{2}x - 2 \) and \( -3x + 2y = 6 \)
2) Find lines that are parallel to the given lines and pass through the given point
   (a) (-2, -3) and \( y = -x + 2 \)

   (b) (-5, -2) and \(-3x + 2y = 6\)

   (c) (-5, 2) and \(-6x + 5y = -10\)

3) Find lines that are parallel to the given lines and pass through the given point
   (a) (-4, 3) and \(-4x + 3y = -6\)

   (b) (3, -5) and \(4x + 9y = -9\)
Challenge Question B

Emma plots the points A(-9,6) and B(-4,4).
She claims that the line AB will be perpendicular to the line $y = 3x - 5$.

Is she correct? Explain your answer.

Assessment Question B

Which of the following lines is not perpendicular to $y = 2x + 1$?

A) $y + \frac{1}{2}x = 6$
B) $2y = 4 - x$
C) $2x + y = 4$
D) $y = -\frac{1}{2}(7 + x)$

Objective C: Midpoints and Ratios on Straight Lines

Finding the midpoint or point from a ratio given two points on a line, the important thing is to think about each coordinate. Finding the midpoint of the $x$-coordinate and the $y$-coordinate for instance is all that needs to be done to get our answer!

Worked Examples C

Worked Example 1

Find the midpoint of the line segment

Diagram NOT accurately drawn
The coordinates are already given, which saves us a job. The important point to note is that in order to find the midpoint of (-1,2) and (7,5) we need only work out the number in the middle (midpoint!) of the \( x \)-coordinates (-1 and 7) and then the number in the middle of the \( y \)-coordinates (2 and 5).

To find the number in the middle to two numbers, we simply sum then and divide by 2:

\[
\begin{align*}
\frac{-1 + 7}{2} &= \frac{6}{2} = 3 \\
\frac{2 + 5}{2} &= \frac{7}{2} = 3.5
\end{align*}
\]

So the midpoint of the line segment is (3, 3.5)

**Worked Example 2**

Find the point C so that AC and BC are the same length

“The same length” is our clue to look for midpoints again, though this time we have to read the coordinates first. Taking care to put the \( x \)-coordinates first and \( y \)-coordinates second, we identify A(4, 2) and B(-8, -1).

Now we again find the middle of the \( x \)-coordinates and the middle of the \( y \)-coordinates:

\[
\begin{align*}
\frac{4 + (-8)}{2} &= \frac{-4}{2} = -2 \\
\frac{2 + (-1)}{2} &= \frac{1}{2} = 0.5
\end{align*}
\]

So (-2, 0.5) is the point C.

Note: we should always check! We can “eyeball” the middle of the line to be about here:

This gives us a good indication that the \( x \)-coordinate should be negative and the \( y \) should be positive, so we can at least be sure we got our signs right!

**Worked Example 3**

Find the point C such that the ratio BC : AC is 2:3
Now we have to use a ratio to find the point on the line, but the process is similar to that with mid-points. First we identify the coordinates A(4, 2) and B(-8, -1).

The difference appears now: we have to think about ratios. Our ratio here indicates that we have 5 parts, and from B to C we have 2 parts and from C to A we have 3 parts.

So we need to find how long BA is, split it into 5 parts and work out where C is. Instead of working with the line BA, we can work with coordinates again! We find the lengths by subtracting the coordinates:

\[
\begin{align*}
    x \text{ length} &= 4 - (-8) = 12 \\
    y \text{ length} &= 2 - (-1) = 3
\end{align*}
\]

(we can count squares and double check with the figure to make sure this makes sense)

Now we need to share these lengths by the ratio: C is 2/5 along the line:

\[
\begin{align*}
    x \text{ length to } C &= \frac{2}{5} \times 12 = \frac{24}{5} = 4.8 \\
    y \text{ length to } C &= \frac{2}{5} \times 3 = \frac{6}{5} = 1.2
\end{align*}
\]

These are lengths however! We need to add them to our start point (coordinate B) to find the coordinates of C.

\[
\begin{align*}
    x \text{ coordinate of } C &= 4.8 + (-8) = -3.2 \\
    y \text{ coordinate of } C &= 1.2 + (-1) = 0.2
\end{align*}
\]

So C has coordinates (-3.2, 0.2), which a quick check of our sketch above seems to confirm.

**Practice Questions C**

1) M is the midpoint of the line segment AB. Find the coordinates of M.
2) N is the midpoint of the line segment CD. Find the coordinates of N.

3) Point E has coordinates (1,4) and point F has coordinates (34,28). Point G lies on the line segment EF such that $\text{EG:GF} = 1:2$. Find the coordinates of point G.

**Challenge Question C**

Using the diagram, find a formula for the point $p$ such that the ratio $Bp : pA$ is $n:m$. 
Assessment Question C

Find the midpoint of the line segment joining the points \( P_1 = (1,3) \) and \( P_2 = (6,8) \)

A) \( \left( \frac{11}{2}, \frac{7}{2} \right) \)

B) \( \left( \frac{7}{2}, \frac{11}{2} \right) \)

C) \((-5,5)\)

D) \((7,11)\)
Lesson 2 – Plotting and Interpreting Straight Line Graphs

Objectives this Tutorial
A) Plotting from $y = mx + c$
B) Finding the equation of a line given points/gradient
C) Solving simultaneous equations with graphs

Starter
Try to think how you might solve the following problems:
• Write down the equation of 3 lines that contain the coordinate (4, -6)
1)  
2)  
3)  
• The gradient between (5, -7) and (a,b) is 2. Find 3 possibilities for (a,b)
1)  
2)  
3)

Objective A: Plotting from $y = mx + c$
When plotting linear graphs we end up with straight lines. The key steps in plotting a linear graph are
(1) Make sure the line is in the correct form ($y = mx + c$)
(2) Work out the range of $x$ values to plot
(3) Draw a table of values
(4) Substitute the $x$ values into the line equation to find the $y$ values
(5) Choose appropriate scales for the axes to plot the data
(6) Plot the points on the graph
(7) Draw the line

Worked Examples A

**Worked Example 1**
Plot $y = 2x + 1$ on the axes given

Following the steps outlined above:
(1) The line $y = 2x + 1$ is already in the correct form
(2) The $x$-axis runs from 0 to 5, so 0, 1, 2, 3, 4, 5 would make the most sense

(3) We draw the table of values

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4) Substitute the $x$ values into the equation $y = 2x + 1$ one by one

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$2 \times 0 + 1 = 1$</td>
<td>$2 \times 1 + 1 = 3$</td>
<td>$2 \times 2 + 1 = 5$</td>
<td>$2 \times 3 + 1 = 7$</td>
<td>$2 \times 4 + 1 = 9$</td>
<td>$2 \times 5 + 1 = 11$</td>
</tr>
</tbody>
</table>

(5) Here the axes are already pre-labelled, so there’s no need to add labels to either one

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible

(7) And finally draw the line with a ruler
This question is a little more complex – we need to careful with negatives and think a bit with part (b), but we can still follow the steps!

(1) If we write the right-hand side of the equation as $-\frac{1}{2}x + 3$ we see the correct form
(2) The $x$-axis runs from -3 to 3, so -3, -2, -1, 0, 1, 2, 3 would make the most sense
(3) We draw the table of values

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4) Substitute the $x$ values into the equation $y = -\frac{1}{2}x + 3$ one by one (being careful with negatives!)

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$\frac{-1}{2} \times -3 + 3 = 4.5$</td>
<td>$\frac{-1}{2} \times -2 + 3 = 4$</td>
<td>$\frac{-1}{2} \times -1 + 3 = 3.5$</td>
<td>$\frac{-1}{2} \times 0 + 3 = 3$</td>
<td>$\frac{-1}{2} \times 1 + 3 = 2.5$</td>
<td>$\frac{-1}{2} \times 2 + 3 = 2$</td>
<td>$\frac{-1}{2} \times 3 + 3 = 1.5$</td>
</tr>
</tbody>
</table>

(5) Here the $x$-axes are already pre-labelled, so we just need to choose a scale for the $y$-axis. Our $y$ values are between 1.5 and 4.5, so 0, 1, 2, 3, 4, 5, 6 is ideal (note: sometimes going up in 2s or 5s etc might make more sense!)

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible
(7) And finally draw the line with a ruler

Now for part (b): we draw a line from \( y = 4.2 \) (as best we can given our scale) to the line we’ve drawn, then vertically downwards to see the corresponding value of \( x \): it is indeed negative, around \(-2.4\).

Worked Example 3

Graph the line described by the equation \( x + y = 3 \) between \( x = -10 \) and \( x = 10 \)

This question requires some thought to save some time, but otherwise more of the same. Following the steps outlined above:
(1) Subtracting \( x \) from both sides gives the correct form: \( y = -x + 3 \) (writing the \( x \) first helps to identify the gradient).
(2) The question asks for \( x \) values between \(-10\) and \(10\), but plotting \(-10, -9, -8\) all the way to \(10\) would waste a lot of time. We choose a suitable increment: \(-10, -5, 0, 5, 10\) for example (when choosing increments, you should plot a minimum of four points to make sure you haven’t made any mistakes!)
(3) We draw the table of values

<table>
<thead>
<tr>
<th>x</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4) Substitute the $x$ values into the equation $y = -x + 3$ one by one

<table>
<thead>
<tr>
<th>x</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>$-(-10) + 3 = 13$</td>
<td>$-(-5) + 3 = 8$</td>
<td>$-(0) + 3 = 3$</td>
<td>$-5 + 3 = -2$</td>
<td>$-10 + 3 = -7$</td>
</tr>
</tbody>
</table>

(5) Our axes are pre-labelled – note that the $x$- and $y$- scales are different. This is not a problem – the only thing to be careful of is to make sure the increment is consistent (2, 4, 6,…. not 2, 4, 7 and with equal size measured on the graph!)

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible

(7) And finally draw the line with a ruler

**Practice Questions A**

1)
   a) Complete the table of values for $y = 2x + 5$

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
b) On the grid draw the graph of \( y = 2x + 5 \) for values of \( x \) from \( x = -2 \) to \( x = 2 \).

2) On the grid draw the graph of \( y = 3x - 2 \) for values of \( x \) from \(-1\) to \(3\).
3) On the grid draw the graph of $x + y = 5$

Challenge Question A

On the grid below, draw appropriate axes and plot $3y = 2(3x - 9)$

Assessment Question A

Which of these lines has the equation $y = 4 - 2x$?
Objective B: Finding the Equation of a Line Given Points or Gradient

We can find all the information about a line (and therefore draw the line) if we have two pieces of information. There are two options here:

Either

4) The coordinates of a point on the line and the gradient of the line

Or

5) The coordinates of two different points on the line

Both cases work similarly, though the second takes an extra step.

If we know the gradient and a point, we can write \( y = mx + c \) and simply substitute in the coordinates of the point and rearrange to find \( c \).

If we know the coordinates of two points on the line, we must first use the formula of the gradient

\[
    m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

to find the gradient, then we write \( y = mx + c \) and as before substitute in either of the given points to find \( c \).

**Worked Examples B**

**Worked Example 1**

Give the equation of the straight line with gradient 2 than passes through (1, 4)

Since we know that the gradient is 2, we know the line must be of the form \( y = 2x + c \).

Since we also know that the line passes through (1, 4), we can find \( c \) by substituting in \( x = 1 \) and \( y = 4 \) and solving for \( c \).

\[
    4 = 2 \times 1 + c \\
    4 = 2 + c \\
    2 = c
\]

So the equation of the line is \( y = 2x + 2 \).

**Worked Example 2**

A line passes through (1, 2) (2,8). What is the equation of the line?

With no gradient given, we need to find that for ourselves.
Using the formula

\[
    m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

We need to identify our two points:

\((1,2)\) and \((2,8)\)  
\((x_1,y_1)\) and \((x_2,y_2)\)

so the gradient is

\[
    m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 2}{2 - 1} = \frac{6}{1} = 6
\]

Now we know that the line must have equation \( y = 6x + c \) and we can find \( c \) just like in the previous example: substitute in one of the points. \((1,2)\) is simpler than \((2,8)\) so let’s choose that one:

\[
    2 = 6 \times 1 + c
\]
so the equation of the line is

\[ y = 6x - 4 \]

**Worked Example 3**

A line passes through (-1, -4) and crosses the y-axis at -2. What is the equation of the line?

This question is almost identical to example 2, but in disguise. We need to recall that if the lines crosses the y-axis at -2, this coordinate is (0, -2) (since every point on the y-axis has x-coordinate 0). Equipped with this, we do the same steps as before:

Using the formula

\[ m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \]

We need to identify our two points:

(-1, -4) and (0, -2)

so the gradient is

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (-4)}{0 - (-1)} = \frac{2}{1} = 2 \]

Now we know that the line must have equation \( y = 2x + c \) and since the question directly tells us the y-intercept is -2, the equation of the line is

\[ y = 2x - 2 \]

**Practice Questions B**

1) A line has gradient of 1.5 and passes through (4, 5). What is the equation of the line?

2) What is the equation of the straight line that passes through (1, -4) and (3, 2)?
3) What is the equation of the straight line that passes through (-5, 2) and (7, -1)?

4) Does the line $y = 3x + 2$ go through (-2, -3)? How do you know?

Challenge Question B

Does the point (0.7, 1.65) lie on the line with gradient $-\frac{1}{2}$ and $x$-intercept 4?

Assessment Question B

A line has a gradient of 4 and passes through the point (1,7). What is its equation?

A) $y = 7x + 4$

B) $y = 4x + 7$

C) $y = 4x + 6$

D) $y = 4x + 3$
Objective C: Solving Simultaneous Equations with Graphs

Every point on a line is a point that satisfies the line’s equation $y = mx + c$. If we plot two straight lines, we see clearly that there is only one crossing point (unless the lines are parallel!). This is true of any straight lines – try it!

If all the red points (forming the red line) satisfy $y = 2x + 2$ and all the blue points forming the blue line satisfy $y = x - 1$, the only point satisfying both equations is (-3, -4) as it lies on both lines.

We can therefore solve simultaneous equations (pairs of equations that are true at the same time) using graphs by looking for the crossing points, though we often have to estimate the solution as it may not lie on a neat grid point.

Worked Examples C

Since the graph lines are already drawn, this is almost entirely a comprehension exercise: recall that all the points on the red line satisfy $y = 6 - x$ and all the points on the blue line satisfy $y = x + 4$ so the only point satisfying both is the point of intersection (1,5). In other words, the simultaneous equations are solved by $x = 1$ and $y = 5$. 
Here we are given one of the lines drawn but not the other. A quick look at the gradients tells us that the line already plotted must be \(2y = -2x + 8\) as \(y = 2x + 1\) has a positive gradient unlike the already plotted line.

In order to plot \(y = 2x + 1\), we can construct a quick table of values (remember, we need at least four points to get a reliable line, and preferably not all next to each other) \(x = -4, 0, 4\) and 8 should do nicely.

<table>
<thead>
<tr>
<th>(x)</th>
<th>-4</th>
<th>0</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>2 (-4) + 1 = (-7)</td>
<td>2 (0) + 1 = (1)</td>
<td>2 (4) + 1 = (9)</td>
<td>2 (8) + 1 = (17)</td>
</tr>
</tbody>
</table>

Studying our graph, it seems that these points won’t fit! Let’s try some points closer to the origin: \(x = -1, 0, 1, 2\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>2 (-1) + 1 = (-1)</td>
<td>2 (0) + 1 = (1)</td>
<td>2 (1) + 1 = (3)</td>
<td>2 (2) + 1 = (5)</td>
</tr>
</tbody>
</table>

Much better - now we plot these points on the grid and joining them with a ruled line:

And we can read off that the point of intersection is \((1, 3)\) so the simultaneous equations have solution \(x = 1, y = 3\).
Practice Questions C

1) Using the graph, write down the solutions to the following pairs of simultaneous equations

A) \( y = 2x + 4 \)
   \( y = x \)

   \( x = \)
   \( y = \)

B) \( y = x \)
   \( 2x + y = 12 \)

   \( x = \)
   \( y = \)

C) \( y = 2x + 4 \)
   \( 2x + y = 12 \)

   \( x = \)
   \( y = \)

2) Using the graph, write down the solutions to the following pairs of simultaneous equations

A) \( x + y = 3 \)
   \( 2x + y = 4 \)

   \( x = \)
   \( y = \)

B) \( x + y = 3 \)
   \( y = x - 5 \)

   \( x = \)
   \( y = \)

C) \( 2x + y = 4 \)
   \( y = x - 5 \)

   \( x = \)
   \( y = \)
3) Plot the two lines and hence solve the equations:
A) $y = 3x - 2$ and $y = x - 2$

Point of intersection: (______, ______)
so $x = _____$ and $y = _____$

B) $x + y = 5$ and $y = 2x - 1$

Point of intersection (______, ______)
so $x = _____$ and $y = _____$

**Challenge Question C**

Two companies are set up at the same time. The first company sells 8 TVs on their first day, and five TVs every two days from then on. The second company sells 4 TVs on their first day, and sell three TVs per day from then on.

Show on a graph of sales how many days it is until the second company's sale total exceeds the first.
Assessment Question C

The lines with the following equations are shown on the graph.

\[ y = 2x - 4 \quad x + y = -10 \]

\[ y = x \quad 2x + 3y = 36 \]

Write down the solutions to the following pairs of simultaneous equations:

a) \( y = 2x - 4 \) and \( y = x \)

\[ x = \]
\[ y = \]

b) \( y = 2x - 4 \) and \( 2x + 3y = 36 \)

\[ x = \]
\[ y = \]

c) \( y = x \) and \( x + y = -10 \)

\[ x = \]
\[ y = \]

d) \( y = 2x - 4 \) and \( x + y = -10 \)

\[ x = \]
\[ y = \]
Lesson 3 – Quadratic Graphs

Objectives this Tutorial
A) Plotting a quadratic graph from an equation  
B) Interpreting quadratic graphs to find roots and values  
C) Approximating minima and lines of symmetry using graphs

Starter
Try to think how you might solve the following problems:
• Write down the equation of 3 quadratic curves that cross the x-axis at (6,0)
  1)  
  2)  
  3)  
• Write down the equation of 3 quadratic curves with a line of symmetry x = 5
  1)  
  2)  
  3)  

Objective A: Plotting a Quadratic Graph from an Equation
When plotting a quadratic graph (an equation that contains an \( x^2 \) term) we end up with a parabola – the bowl shape plotted here:
The key steps in plotting a quadratic graph is the same as with linear graphs:
(1) Get the equation is in the correct form \( y = ax^2 + bx + c \)  
(2) Work out the range of \( x \) values to plot  
(3) Draw a table of values  
(4) Substitute the \( x \) values into the line equation to find the \( y \) values (be careful with negatives!)  
(5) Choose appropriate scales for the axes to plot the data  
(6) Plot the points on the graph  
(7) Draw the line

Worked Examples A

Worked Example 1
(a) Complete the table of values for \( y = x^2 + x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) On the grid, draw the graph of \( y = x^2 + x \)

Following the steps outlined above:
(1) The equation \( y = x^2 + x \) is already in the correct form (with no constant \((c)\) term)

(2) We’ve been given the table of values already so we know our \( x \) values

(3) We draw the table of values

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>((−3)^2 + (−3)) = 6</td>
<td>((−2)^2 + (−2)) = 2</td>
<td>((−1)^2 + (−1)) = 0</td>
<td>((0)^2 + (0)) = 0</td>
<td>((1)^2 + (1)) = 2</td>
<td>((2)^2 + (2)) = 6</td>
<td>((3)^2 + (3)) = 12</td>
</tr>
</tbody>
</table>

(4) Substitute the \( x \) values into the equation \( y = x^2 + x \) one by one. CAUTION: when substituting \( x = −3 \) into \( x^2 + x \), a very common mistake is to write

\[
-3^2 + -3 = -9 - 3 = -12
\]

This is not correct! Remember, when substituting, we should include brackets to be safe!

\[
(-3)^2 + (-3) = 9 - 3 = 6
\]

(5) Here the axes are already pre-labelled, so there’s no need to add labels to either one

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible

Hint: if your points don’t look like they’re in a parabola shape, check your negative substitutions for mistakes!

(7) And finally draw the line – here’s another key difference: because we know we’re getting a parabola, which is curved, we **don’t use a ruler**! Using a pencil will help you to get a curve that goes through the points and has the right shape

Note the bowl-shape and the symmetry of the graph!
Following the steps outlined above:

1. The equation \( y = x^2 - 2x - 4 \) is already in the correct form.
2. We’ve been given the table of values already so we know our \( x \) values.
3. We draw the table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>[ (-2)^2 - 2(-2) - 4 = 4 + 4 - 4 = 4 ]</td>
<td>[ (-1)^2 - 2(-1) - 4 = 1 + 2 - 4 = -1 ]</td>
<td></td>
<td>[ (0)^2 - 2(0) - 4 = 0 + 0 - 4 = -4 ]</td>
<td>[ (1)^2 - 2(1) - 4 = 1 - 2 - 4 = -5 ]</td>
<td>[ (2)^2 - 2(2) - 4 = 4 - 4 - 4 = -4 ]</td>
<td>[ (3)^2 - 2(3) - 4 = 9 - 6 - 4 = -1 ]</td>
</tr>
</tbody>
</table>

4. Substitute the \( x \) values into the equation \( y = x^2 - 2x - 4 \) one by one. Again, be careful with negatives!

5. Here the axes are already pre-labelled, so there’s no need to add labels to either one. Note that the scale is different on each axis!

6. Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible.

Hint: if your points don’t look like they’re in a parabola shape, check your negative substitutions for mistakes!
(7) And finally draw the line – here’s another key difference: because we know we’re getting a parabola, which is curved, we **don’t use a ruler**! Using a pencil will help you to get a curve that goes through the points and has the right shape.

For part (c), we can see from the graph that the lowest value that the graph reaches is -5; this is the minimum value of \( y \), since the graph keeps climbing in both directions (since it’s a parabola!)

(Note: this is related to the fact that we can complete the square – as an extension, complete the square for this function and see how we can find the minimum value of \( y \) that way):

**Worked Example 3**

Plot the function \( f(x) = x^2 - 4 \) on the axes shown

This question looks different, but the aim is the same: plotting a quadratic graph. Highlighting the differences

- We have \( f(x) \) instead of \( y \). As long as we label our ‘\( y \)’ axis \( f(x) \) this doesn’t change anything in the question – it’s like a different label or name for the graph, but nothing else
- There is no __\( x \) term in the equation: again, this doesn’t matter. As long as there is any kind of __\( x^2 \) term, we’re still plotting a quadratic, so the usual rules apply
- We aren’t provided with a table of values: then we draw our own!

Once again, following the steps outlined above:

1. The equation \( f(x) = x^2 - 4 \) is already in the correct form
2. The \( x \)-values are clearly -2, -1, 0, 1, 2, 3
3. We draw the table of values
(4) Substitute the $x$ values into the equation $y = x^2 - 4$ one by one. Again, be careful with negatives!

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$(-2)^2 - 4$</td>
<td>$(-1)^2 - 4$</td>
<td>$(0)^2 - 4$</td>
<td>$(1)^2 - 4$</td>
<td>$(2)^2 - 4$</td>
<td>$(3)^2 - 4$</td>
</tr>
<tr>
<td></td>
<td>$= 4 - 4$</td>
<td>$= 1 - 4$</td>
<td>$= 0 - 4$</td>
<td>$= 1 - 4$</td>
<td>$= 4 - 4$</td>
<td>$= 9 - 4$</td>
</tr>
<tr>
<td></td>
<td>$= 0$</td>
<td>$= -3$</td>
<td>$= -4$</td>
<td>$= -3$</td>
<td>$= 0$</td>
<td>$= 5$</td>
</tr>
</tbody>
</table>

(5) Here the axes are already pre-labelled, so there’s no need to add labels to either one. Note that the scale is different on each axis!

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible. 

Hint: if your points don’t look like they’re in a parabola shape, check your negative substitutions for mistakes!

(7) And finally draw the line – here’s another key difference: because we know we’re getting a parabola, which is curved, we **don’t use a ruler!** Using a pencil will help you to get a curve that goes through the points and has the right shape.
Practice Questions A

(1) (a) For each equation, complete the table of values.
(b) Graph the function on the axes shown, labelling each one.

\[ y = x^2 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ y = x^2 + 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ y = x^2 - 3 \]

<table>
<thead>
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<th>x</th>
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<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ y = x^2 + 2x \]

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ y = x^2 - 2x - 3 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ y = 2x^2 - 4x + 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2) On the grid below, draw the graph of \( y = x^2 - 3x - 1 \) and use your graph to estimate the minimum value of \( y \).

(3) On the axes below, graph the function \( f(x) = x^2 - x + 3 \).
Challenge Question A

The height of a cannon ball over time is given by the equation \( h(t) = -3t^2 + 18t \).

(a) Graph the height of the cannonball over time

(b) Find the maximum height of the cannonball

(c) How many seconds after firing does the cannonball achieve maximum height?

Assessment Question A

Which graph shows the plot of the equation \( y = x^2 - 3x + 2 \)?
Objective B: Interpreting Quadratic Graphs to Find Roots and Values

Finding roots is one of the main reasons to plot quadratic equations. A root of a quadratic $ax^2 + bx + c$ is simply a value of $x$ which solves the equation $ax^2 + bx + c = 0$. We already know that we can solve this equation by

- Factorising into double brackets
- Completing the square
- Using the quadratic formula

However, when given a quadratic graph, we don’t need to do any of these.

Recall what drawing a graph actually means: every point on the curve is a solution to the equation $y = ax^2 + bx + c$. If we want the values of $x$ such that $ax^2 + bx + c = 0$, then it should be clear that we need to look at the points where $y = 0$ (i.e. where the graph crosses the line $x = 0$)! The values of $x$ at these crossing points satisfy $y = ax^2 + bx + c$ since they’re on the curve, and the value of $y$ here is 0, so we have our roots by simple observation!

[note: this is actually a sort of special case of a simultaneous equation! We have one line given by our equation $y = ax^2 + bx + c$ and another line given by $y = 0$ and we look for the intersections!]

When finding values rather than roots, the same logic applies, but a question might ask for us to find the value of $x$ such that $x^2 + 2x + 1 = 3$, so we find the intersection of the graph $x^2 + 2x + 1$ with $y = 3$.

**Worked Examples B**

**Worked Example 1**

The figure shows the graph $y = x^2 - 2x - 4$. Using the graph, estimate the roots of the equation.

We can see here that the roots here are $x = -1.25$ and $x = 3.25$

[note: there are some quadratics that have only one root or even no roots (at least that we can see...) can you think of any such a quadratics? Try drawing one, then thinking of an equation!]

Remember, the roots of the equation are the $x$-coordinates of the points where the graph intersects the line $y = 0$ (i.e. the $x$-axis)
Instead of the roots, which means we need to look for the intersections between the graph and the line \( y = 0 \), here we need to find the intersection between the graph and the line \( y = 1.6 \). This looks like an awkward number, but if we look at the scale on our graph it’s not too bad!

The scale on the \( y \)-axis is 5 little ticks = 2 so 1 little tick = \( 2 \div 5 = 0.4 \). So 1.6 is the line just below 2. Drawing that line and tracing the \( x \)-values:

Here we can see that \( x = -1.8 \) and \( x = 0.9 \) are our solutions.

**Practice Questions B**

1) Using the graphs whose equations are indicated in red, find estimates of the solutions to the following equations:

(a) \( x^2 - 2x + 5 = 0 \)

(b) \( x^2 + x - 1 = 0 \)

(c) \( x^2 - 2x - 13 = 0 \)

\[ x = \quad \] \[ x = \quad \] \[ x = \quad \]
2) Using the graphs whose equations are indicated in red, find estimates of the solutions to the following equations

(a) \( x^2 + 2x - 1 = 0 \)
\( x = \) _____
(b) \( x^2 - x + 1 = 0 \)
\( x = \) _____
(c) \( 2x^2 - 5x - 1 = 0 \)
\( x = \) _____

3) The graph of \( y = f(x) \) is drawn on the grid.
   a) Write down the roots of \( f(x) = 0 \)

   b) Use the graph to estimate the roots of \( f(x) = -1 \)

   c) Write down the coordinates of the turning point of the graph

Challenge Question B

The grid below shows the graph of \( y = 2x^2 - 4x + 1 \)
The graph of \( 2x^2 - 4x + 1 = k \) has exactly one solution.
Use the graph to find the value of \( k \).

Assessment Question B

This is the graph of \( y = x^2 - 3x - 2 \).
Using the graph, which of the following answers provides the best approximation to the solution of \( x^2 - 3x - 2 = 3? \)

A) \( x = -1.2, x = 4.2 \)
B) \( x = -0.5, x = 3.5 \)
C) \( x = -2 \)
D) \( x = 1.2, x = -4.2 \)
Objective C: Approximating Minima and Symmetry Using Graphs

As we can see from all the quadratics we’ve plotted so far, we can see that each and every quadratic when plotted gives a parabola.

Every parabola plotted from the form \( y = x^2 + bx + c \) is symmetrical along a vertical line of symmetry (in this figure \( x = 1 \)).

Every parabola plotted from the form \( y = x^2 + bx + c \) has a minimum value (in this figure \( y = -5 \)).

[This is again related to the fact that we can complete the square given a quadratic equation]

We can see these on a plot much easier than completing the square however!

Worked Examples C

**Worked Example 1**

The figure shows the graph \( y = x^2 + x \).

Using the graph, estimate:
(a) The minimum value of \( y \)
(b) The line of symmetry of \( y \)

The minimum value of \( y \) here is tricky to identify but appears to be a little above the first minor gridline below zero. Each minor gridline in the \( y \)-axis is \( 2 \div 5 = 0.4 \), so the minimum value of \( y \) is around \(-0.4\).

For the line of symmetry, it is often best to identify to points with the same \( y \)-value: in this instance, I can see that the graph cuts the \( x \)-axis at 0 and -1. I know that the graph is symmetrical, so my line of symmetry is half way between those points, at \( x = -0.5 \).
This question uses some different language, but for the same maths. For the first part we need to find the line of symmetry, and for the second we need to identify the turning point (the point at which the graph turns – the minimum!). The use of the word “approximate” is used here because we have to draw lines and use our best judgement for the $x$- and $y$-values.

Again, for identifying the line of symmetry we find two points on the curve with the same $y$-value. We can choose the $x$-intercepts, but we can also see that the curve has the $y$-value of -4 at both $x = 0$ and $x = 2$ so half-way between those two gives $x = 1$ as the line of symmetry.

For the turning point, we need to identify the coordinates of the point highlighted in gold; the minimum of the graph. The $y$-coordinate we can estimate on the graph to be around $y = -3$. We already found the $x$-coordinate in finding the axis of symmetry: $x = 1$.

Therefore we can sue the graph to approximate the turning point to be (1, -3)

**Practice Questions C**

(1) The following figure shows five quadratics. By factorising the following equations, identify the curve and find
   a. The minimum value of this graph
   b. The line of symmetry of this graph

\[
y = x^2 - 8x + 7 \quad \text{colour:} \quad \text{minimum:} \quad \text{line of symmetry:}
\]
\[
y = x^2 - 8x + 15 \quad \text{colour:} \quad \text{minimum:} \quad \text{line of symmetry:}
\]
\[
y = x^2 - 5x - 14 \quad \text{colour:} \quad \text{minimum:} \quad \text{line of symmetry:}
\]
\[
y = x^2 + 12x + 35 \quad \text{colour:} \quad \text{minimum:} \quad \text{line of symmetry:}
\]
\[
y = x^2 + 2x - 8 \quad \text{colour:} \quad \text{minimum:} \quad \text{line of symmetry:}
\]
(2) Looking at your answers from the above, can you find a way to predict the line of symmetry for a function? What about the minimum?

(3) Find the coordinates of the turning points for the function $y = x^2 - 5x + 5$ (can be done using the graph above)
Challenge Question C

(a) A quadratic has the form $x^2 + bx + c$. What is the line of symmetry of the quadratic? What is its minimum value?

(b) We need to use different terminology with quadratics of the form $-x^2 + bx + c$. Why does the term minimum no longer make any sense? What word should we use instead?

Assessment Question C

A parabola has a turning point at (-3,2). Where is the axis of symmetry?

A) $y = -2$
B) $x = 3$
C) $x = -3$
D) $y = 2$
Lesson 4 – Advanced Graphs

Objectives this Tutorial
A) Solving quadratic simultaneous equations using a graph
B) Recognising advanced graphs
C) Plotting advanced graphs

Starter
Try to think how you might solve the following problems:
• Write down the equation of 3 graphs that cross the x-axis at (3,0)
  1) 
  2) 
  3) 
• Write down the coordinates of 3 points on the graph of \( y = 4^x \) that all have negative x-values
  1) 
  2) 
  3) 

Objective A: Solving Quadratic Simultaneous Equations Using a Graph

Just as in tutorial 3, solving simultaneous equations where one of the equations is a quadratic equation can be done using graphs simply by looking for the points where the two graphs meet! This is because Each graph is the set of points that solve the equation of that graph, so in order to solve both equations the solution must lie on both graphs (where the lines meet!)

Questions in this area often provide one graph and will ask you to draw “an appropriate line” to find the solutions. The first worked example below has the full breakdown, whereas the second requires us to put the pieces together ourselves.
Worked Examples A

Worked Example 1

Example

a) Draw the graph of \( y = x^2 - 9 \)
b) Explain why your graph shows the solutions to \( x^2 - 9 = 0 \) are -3 and 3
c) By drawing a suitable line find approximate solutions to \( x^2 - 9 = 6 \)
d) Draw the line \( y = 4x - 12 \)
e) Use your answer to part d) to find the solutions to \( x^2 - 9 = 4x - 12 \)

This question combines a lot of the work from our previous tutorials. For part a) we fill in the table (being careful not to muddle up our minus signs!)

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>((-4)^2 - 9) = 7</td>
<td>((-3)^2 - 9) = 0</td>
<td>((-2)^2 - 9) = -5</td>
<td>((-1)^2 - 9) = -8</td>
<td>9</td>
<td>((1)^2 - 9) = -8</td>
<td>((2)^2 - 9) = -5</td>
<td>((3)^2 - 9) = 0</td>
<td>(-6)</td>
</tr>
</tbody>
</table>

Then we plot the points on our graph and remember: quadratics are curves, not straight lines!
For part b), recall that the points on the graph are the points where \( y = x^2 - 9 \) so the points that lie on the line \( y = 0 \) (the x-axis) are our solutions: from the graph we can see that we have 3 and -3 as the question suggests!

The same logic works for part c), but this time we need to draw a line. In order to find the solutions to \( x^2 - 9 = 6 \), we need to draw the line \( y = 6 \) and find the crossing points.

Here we can see that the crossing points are at around \( x = 3.75 \) and \( x = -3.75 \) (NOTE: when dealing with quadratics, we will often have TWO solutions to questions, because of the shape of the graph!)

For part d), we draw a table of values and plot the straight line \( y = 4x - 12 \):

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
</table>
| y  | 4x-4= -12  
=28 | 4x-3=-12  
=24  | 4x-2=-12  
=20  | 4x-1=-12  
=16  | 4x0=-12  
=0  | 4x1=-12  
=-8  | 4x2=-12  
=-4  | 4x3=-12  
=0  | 4x4=-12  
=4  |
And finally we can answer part e): the solutions to the equation $x^2 - 9 = 4x - 12$ are the points where the graphs of $y = x^2 - 9$ and $y = 4x - 12$ cross, which we can see here happens at $x = 1$ and $x = 3$.

**Worked Example 2**

Plotted here is the graph $y = x^2 + 2x + 3$

(a) By drawing a suitable line, solve the equation $x^2 + 2x + 3 = x + 5$

(b) Hence find the solutions to the equation $x^2 + 3x + 2 = 2x + 4$

This question has fewer parts, but it does give us the quadratic graph already which saves us some time!

Part (a) is asking us to solve the equation $x^2 + 2x + 3 = x + 5$: we already have the graph of the left-hand side plotted, so all we need to do is plot the graph $y = x + 5$ and find the crossing points. We don’t need to choose all the $x$-values, but we need a spread:

<table>
<thead>
<tr>
<th></th>
<th>-6</th>
<th>-3</th>
<th>0</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$-6 + 5 = -1$</td>
<td>$-3 + 5 = 2$</td>
<td>$0 + 5 = 5$</td>
<td>$3 + 5 = 8$</td>
<td>$6 + 5 = 11$</td>
</tr>
</tbody>
</table>
Looking at the crossing points, we see the solutions are $x = -2$ and $x = 1$.

Now for the tricky part, part (b): are we supposed to plot another quadratic?! These are tough GCSE questions, but are often a lot more straightforward than they look. One clue here is that this is a part (b), not an entirely new question, so probably involves the graphs we’ve already drawn. Another is the ‘hence’ which means we should be able to us part (a) to solve part (b).

Here’s the trick:

\[
\begin{align*}
+1 - x & \quad x^2 + 3x + 2 = 2x + 4 \\
+1 - x & \quad x^2 + 2x + 3 = x + 5
\end{align*}
\]

So the solutions to part (b) are exactly the same as those to part (a)! A trick on questions like this is to always try to rearrange your new equation so that one side is the quadratic you’ve already drawn. Sometimes the other equation might be different so you might have to draw another straight line, but that’s much easier than trying to get another quadratic!

**Practice Questions A**

1) Plotted is the graph $y = x^2 + x - 1$. By plotting the graph $y = x + 3$, find the solutions to the equation $x^2 + x - 1 = x + 3$
2) By plotting the appropriate lines, show that the solutions of $x^2 - x - 2 = x + 2$ lie in the regions $-2 \leq x \leq -1$ and $3 \leq x \leq 4$. 
3) The graph shows the line $y = x^2 - 3x$. By plotting an appropriate line, estimate the negative solution to the equation $x^2 - 4.5x + 1 = 0$.
**Challenge Question A**

The graph shows the plot of $y = x^3 - 3x^2 + x + 1$. By plotting an appropriate line, find the solutions to the equation $y = x^3 - 3x^2 + 1$.

---

**Assessment Question A**

Here is the graph of $y = x^2 + 3x - 1$.

What line would you need to draw to find an approximate solution to $x^2 + 4x - 4 = 0$?

A) $y = 0$
B) $y = 3 - x$
C) $y = x - 3$
D) $y = -3$
Objective B: Recognising Advanced Graphs

There are of course more graphs than just linear and quadratic (as the challenge question just now revealed!). Here are some more graphs we should be able to recognise.

<table>
<thead>
<tr>
<th>Graph type</th>
<th>Important features</th>
<th>Sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linear graphs</strong></td>
<td>$y = mx + c$ Gradient, intercept Straight line (ruler!)</td>
<td><img src="image1.png" alt="Sketch" /> <img src="image2.png" alt="Sketch" /></td>
</tr>
<tr>
<td></td>
<td>$m$ positive (1) or negative (2)</td>
<td></td>
</tr>
<tr>
<td><strong>Quadratic graphs</strong></td>
<td>$y = ax^2 + bx + c$ a positive (1) or negative (2)? Parabola shape - curve</td>
<td><img src="image3.png" alt="Sketch" /> <img src="image4.png" alt="Sketch" /></td>
</tr>
<tr>
<td><strong>Cubic graphs</strong></td>
<td>$y = ax^3 + bx^2 + cx + d$ a positive (1) or negative (2)? A curve up and down</td>
<td><img src="image5.png" alt="Sketch" /> <img src="image6.png" alt="Sketch" /></td>
</tr>
<tr>
<td><strong>Reciprocal graphs</strong></td>
<td>$y = \frac{1}{x}$ Not connected!</td>
<td><img src="image7.png" alt="Sketch" /></td>
</tr>
<tr>
<td><strong>Exponential graphs</strong></td>
<td>$y = a^x$ Always passes through (0,1)</td>
<td><img src="image8.png" alt="Sketch" /></td>
</tr>
<tr>
<td><strong>(*)Circle graphs</strong></td>
<td>$x^2 + y^2 = r^2$ $r$ is the radius of the circle centred at (0,0)</td>
<td><img src="image9.png" alt="Sketch" /></td>
</tr>
<tr>
<td><strong>(*)Square root graphs</strong></td>
<td>$y = \sqrt{x}$ Half a parabola on its side</td>
<td><img src="image10.png" alt="Sketch" /></td>
</tr>
</tbody>
</table>

* these graphs are used in other areas of maths but we won’t go into depth with them in this book.
Worked Examples B

**Worked Example 1**

Which of these graphs is
(a) A linear graph with positive gradient?
(b) A cubic graph?
(c) An exponential graph?

First of all we have to distinguish between our two straight lines. The one with the positive gradient is the green one because it traces a line from bottom left to top right. (If in doubt, pick a simple line like $y = x + 1$ and do a quick table of values to see what a positive gradient graph would look like!)

Then we have the purple and the red for cubic and exponential. The cubic graph always has an up and a down curve (even if there's a curve in the middle, like here!) so red is the cubic curve.

This leaves purple to be the exponential, which looks good because exponential graphs are of the form e.g. $2^x$. Importantly this can't be negative for any value of $x$ (try it and see!) so the purple line makes sense.

**Worked Example 2**

Match the graphs with the equations

1. $y = 2x$
2. $y = 2x^2$
3. $y = 2x^3$
4. $y = \frac{2}{x}$

This question is nice and straightforward, but can confuse us with the coefficients in the equations.

More important than the coefficients are the overall shapes! For example equation 1 is clearly a straight line graph – there is only an $x$ term, no powers etc – only a gradient. The only straight line option for us is graph b).

Similarly graph a) is the only quadratic graph (the give-away parabola bowl shape!) and the only option that’s quadratic is equation 2.

The other two: one cubic ($x^3$) and one reciprocal ($\frac{1}{x}$) graph. A good tip here is to think that $x$, $x^2$ and $x^3$ all follow a pattern, where $\frac{1}{x}$ is the odd one out – the graphs also have an odd one out – graph c is broken into two pieces! So equation 3 is graph d and equation 4 is graph c.
Practice Questions B

(1) Match the equation to the graph

\[
\begin{align*}
  y &= 3x + 2 \\
  y &= 2x + 3 \\
  y &= x^2 + 3 \\
  y &= -x^2 + 3 \\
  y &= x^3 + 3 \\
  y &= -x^3 + 3 \\
  y &= x^3 - x^2 + 3 \\
  y &= x^2 + 2x + 3 \\
  y &= \frac{1}{x} \\
  y &= 2^x
\end{align*}
\]
(2) Match each of the seven graphs to one of the seven equations below. You must justify your answer!

Which equation? 
Why?

Which equation? 
Why?

Which equation? 
Why?

Which equation? 
Why?

Which equation? 
Why?

Which equation? 
Why?

Which equation? 
Why?

\begin{align*}
y &= 3x + 5 \\
y &= x^2 + 2 \\
y &= \frac{1}{x} \\
y &= -2x + 3 \\
y &= x^3 + 2 \\
y &= -x^2 + 5 \\
y &= 2^x
\end{align*}
Challenge Question B

Match the equation to the graph

\[
\begin{align*}
y &= x^3 \\
y &= 2x + 1 \\
y &= \frac{1}{x} \\
y &= x^2 - 1 \\
y &= -2x - 1 \\
y &= x^2 \\
y &= 2x - 1 \\
y &= -x^3 \\
y &= x^2 + 1 \\
y &= 1 - 2x \\
y &= (x - 1)^2 \\
y &= -x^2
\end{align*}
\]

Assessment Question B

What is the equation of the curve shown plotted here?

A) \( y = 5^x \)

B) \( y = x^2 + 1 \)

C) \( y = \frac{5}{x} \)

D) \( y = 5x^3 + 1 \)
Objective C: Plotting Advanced Graphs

Plotting advanced graphs may look intimidating but is no more difficult than plotting any other kind of graph! The important thing to have in mind is the shape you’re expecting out the end so you can check that your plot makes sense when you’re done.

As before, plotting a graph is easiest with a table of values, then we can follow the same instructions as before:

1. Make sure the equation is in the correct form \( y = f(x) \), in other words all the \( x \)s are on one side.
2. Work out the range of \( x \) values to plot.
3. Draw a table of values.
4. Substitute the \( x \) values into the line equation to find the \( y \) values (be careful with negatives!)
5. Choose appropriate scales for the axes to plot the data.
6. Plot the points on the graph.
7. Draw the line – remember if the line is not a linear graph it should be a curve.

Worked Examples C

**Worked Example 1**

Draw the graph of

\[ y = x^3 + 4x^2 + x + 1 \]

For values of \( x \) between -4 and 2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(-4^3 + 4(-4)^2 + (-4) + 1)</td>
<td>(-3^3 + 4(-3)^2 + (-3) + 1)</td>
<td>(-2^3 + 4(-2)^2 + (-2) + 1)</td>
<td>(-1^3 + 4(-1)^2 + (-1) + 1)</td>
<td>(0^3 + 4(0)^2 + (0) + 1)</td>
<td>(1^3 + 4(1)^2 + (1) + 1)</td>
<td>(2^3 + 4(2)^2 + (2) + 1)</td>
</tr>
<tr>
<td></td>
<td>-3</td>
<td>7</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>27</td>
</tr>
</tbody>
</table>

Then we plot our points:

And when connecting the points, we have to think about the overall shape we should be aiming for: a nice smooth curve with a wiggle in the middle.

(again, if in doubt, say ‘what happens between -3 and -2? – substitute in \( x = -2.5 \) and see!)

![Graph of y = x^3 + 4x^2 + x + 1 for x between -4 and 2.](image)
**Worked Example 2**

Draw the graph of

\[ y = \frac{4}{x} \]

Between \( x = -5 \) and 5.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>4</td>
<td>-4</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>4</td>
<td>2.5</td>
<td>1.5</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>-0.8</td>
<td>-1</td>
<td>-1.33..</td>
<td>-2</td>
<td>-4</td>
<td>!!!</td>
<td>4</td>
<td>2</td>
<td>1.33..</td>
<td>1</td>
<td>0.8</td>
</tr>
</tbody>
</table>

A big problem here: there is no value for \( x = 0 \)! And we know that there can’t be since we can’t divide by zero ('how many zeroes go into 4' doesn’t even make sense as a question!) this is a good indicator and confirmation of our reciprocal graph where we have the two different parts, but let’s plot and see:

Again, with the points plotted the image we should be thinking of is the two slopes as seen in the previous section, but if we want to confirm that we can put in some more points – e.g. \( x = \frac{1}{2}, x = -\frac{1}{2} \) might help!

And again we draw the curve (not straight lines).
Practice Questions C

(1) Plot the graph of $y = x^3 + 2x^2 - 3x + 1$ for values of $x$ between -4 and 2.

(2) Draw the graph of $y = 4^x$ for $x$ values between -2 and 2.

(3) Plot the graph $y = \frac{-2}{x} + 1$ for $x$ values between -5 and 5.
Challenge Question C

At day $t = 0$ a radioactive material has mass 10kg. The material decays and loses mass radioactively by the equation $m = 10e^{-0.5t}$ where $e$ is Euler’s number (see your calculator $e^x$). By plotting this function, find out how much of the mass is left after 4 days.

Assessment Question C

This is a table of values for $y = x^3 - 3x + 1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td>⋆</td>
<td></td>
<td></td>
<td>-2</td>
</tr>
</tbody>
</table>

What value should replace the star?

A) -11  
B) 1  
C) -1  
D) -13
Objectives this Tutorial

A) Revise any insecure topics
B) Sit the post-test

Objective A:

Look back through the contents page of this book and note any topics you feel less confident with. Spend the first part of this tutorial revising this material using the available resources.
Reflecting on this booklet

<table>
<thead>
<tr>
<th>What did you most enjoy about working through this programme?</th>
</tr>
</thead>
<tbody>
<tr>
<td>•</td>
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<tr>
<td>•</td>
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<tr>
<td>•</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>What did you find challenging about the programme?</th>
<th>How did you overcome these challenges?</th>
</tr>
</thead>
<tbody>
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<td>•</td>
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<td>•</td>
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<td>•</td>
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</tr>
</tbody>
</table>
A Barrel of Graphs
Key Stage 4 Programme

researchersinschools.org
Lesson 1: Straight Line Graphs

Objectives this Tutorial
A) Understanding \( y = mx + c \)
B) Being able to find midpoints and ratios on lines
C) Being able to find and recognise parallel and perpendicular lines

Starter
Try to think how you might solve the following problems:
- Find 3 possible pairs of coordinates so that \((-3,7)\) is the midpoint
  1)
  2)
  3)
- Write down the equations of 3 lines parallel to \(3y + x = 7\)
  1)
  2)
  3)

Objective A: Understanding \( y = mx + c \)

Every linear graph (every graph of a straight line) can be written in the form

\[ y = mx + c \]

where \( x \) and \( y \) are variables; quantities that vary depending on one another, such as ‘height and weight’ of a person or ‘age and cost’ of a phone. These appear on the axes of the graph. The other quantities \( m \) and \( c \) are called the gradient and \( y \)-intercept respectively. The \( y \)-intercept is the easiest to explain: it’s just the point where the straight line crosses. The gradient gives information about how steep the line is.

But what does the straight line graph actually mean? \( y = mx + c \) tells us that if we take a point \( x \), multiply it by \( m \) and add \( c \), we get \( y \). For example, \( y = 2x + 1 \) with \( x = 1 \) yields \( y = 3 \). This tells us that \((1,3)\) is a solution of the equation. But also if \( x = 0 \) then \( y = 1 \) so \((0,1)\) is also a solution to the equation. It turns out that there are a lot of solutions to equations like this, and they lie on a straight line! The line we draw marks out every possible solution of the equation. Points that don’t lie on the line just denote \( x \),\( y \) pairs that don’t ‘match up’ in the given equation.

The gradient of a straight line is defined by the formula

\[ \text{gradient} = \frac{\text{change in } y}{\text{change in } x} \]

Or if the graph is know to go through coordinates \((x_1,y_1)\) and \((x_2,y_2)\)

\[ \text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} \]
In this formula, note it doesn’t matter which coordinate pair we call the $x_1$ and which we call the $x_2$ – we get the same result either way (try it!).

Useful facts to remember about the gradient:

- A positive gradient slopes upward (from bottom left to top right: see the green line)
- A negative gradient slopes downward (from top left to bottom right: see the red line)
- The size of the gradient tells us how steep the slope is: the bigger, the steeper.

**Worked Examples A**

**Worked Example 1**

Write down the gradients and intercepts of the following lines

(a) $y = 12 - 12x$

(b) $x + 2y = 2$

For this question we need to remember that when we rearrange our line to be in the form $y = mx + c$ then we can read off the gradient and intercept easily!

For the first part, we can write the equation as $y = -12x + 12$ (make sure to take care with the signs!) and then we can read off the gradient and intercept:

- gradient = $-12$
- intercept = $12$

For the second line, a little more work is needed: first to isolate the $y$ term by subtracting $x$ from both sides $x + 2y = 2$

$$2y = 2 - x$$

Then to get into the right form we must divide both sides by $2$ (be careful to divide each term on both sides!)

$$y = 1 - \frac{x}{2}$$

Which we can again rearrange to make it clearer to read off the gradient and intercept

- gradient = $-\frac{1}{2}$
- intercept = $1$

**Worked Example 2**

(a) Work out the gradient of the line $AB$.

P is the point $(6, -2)$.

(b) Work out the gradient of the lines $AP$ and $PB$.
In this question we’ve been given two coordinates already. For part (a) we can simply read the difference in the $y$-coordinates by subtracting the $y$-coordinate of A from the $y$-coordinate of B and the same for the $x$-coordinates:

$$m = \frac{5 - 2}{7 - (-1)} = \frac{3}{8}$$

And we leave it positive since the graph is from bottom left to top right.

For the second part, it helps to mark on the question approximately where the point is. We can see from the coordinates which quadrant (quarter) of the graph it must be in, and we know it must be to the left of the point B (since it has a smaller $x$-coordinate)

Then we repeat the procedure above:

For the line AP

$$m = \frac{2 - 2}{-1 - 6} = -\frac{4}{5}$$

And as the line must be drawn from upper-left to lower-right the negative sign is correct.

Note: if the numbers have been entered into the formula correctly, we can circle them. If done incorrectly, they won’t be in a column.

For the line PB:

$$m = \frac{5 - 2}{7 - 6} = \frac{7}{1} = 7$$

And as the line must be drawn from lower-left to upper-right the no negative sign is needed!

**Worked Example 3**

Find

(a) the gradient
(b) the $y$-intercept and
(c) the equation of the line in the figure

Step 1: pick two exact points

In order to get an accurate value for the gradient, two points on the graph are needed. We can make life much easier for ourselves by choosing the two points wisely. Here we can notice that our line is on a grid-line when $x = 1$ ($y = 10$) and when $x = 6$ ($y = 50$) so we can choose those as our two points, rather than struggle with decimals and guessing between grid-lines.
Step 2: Find the gradient

There are two methods for finding the gradient

**Method 1**

The easiest way to implement this is to draw the right-angled triangle between the two points to find the change in $x$ and the change in $y$.

Then we need to look at the direction of the slope for the sign of the gradient: it’s directed from bottom left to top right, so must be positive.

**Method 2**

Use the formula: let’s choose $(1,10)$ for the $(x_1,y_1)$ and $(6,50)$ for the $(x_2,y_2)$ (again note we could have chosen the other way around, it doesn’t matter! It only matters that we don’t mix up and use one of our points for $x_1$ and the other point for $y_1$)

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{50 - 10}{6 - 1} = \frac{40}{5} = 8$$

Note: a gradient of 8 is a steep line (for every 1 step in $x$ the line goes 8 in $y$)! The reason it doesn’t look so steep on the graph is because the scale of the axes are different for $x$ and $y$!

Step 3: find the $y$-intercept

Simply read the value from the $y$-axis where it touches the green line (and be careful with the scale – in this instance the major grid-line on the $y$-axis is 10, so a minor one is 2. (This isn’t always true on all graph scales, count to see how many minor grid-lines are in a major one!)

Step 4: write the equation of the line

We’ve already found the gradient and the intercept, so all that’s left is to write $y = mx + c$ with our gradient $m$ and our intercept $c$:

$$y = 8x + 2$$

**Practice Questions A**

1) Find the gradient of the graph given by the green line shown

\[ -\frac{1}{2} \]
2) Find the gradient of the graph given by the blue line shown

3) Write down the gradient of the straight line with equation $3y = 2(3 - 2x)$

$$y = 2 - \frac{4}{3}x$$

$$-\frac{4}{3}$$

4) Write down the gradient of the straight line with equation $2 = 2y + 3x$

$$-\frac{3}{2}$$
Challenge Question A

A scientist adjusts the current while measuring the voltage across an electronic device.

(a) The scientist plots a best-fit line. Find the gradient and y-intercept of this line.

(b) Given that Voltage ($V$) is related to the device’s resistance ($R$) and current ($I$) flowing through the circuit by the formula $V = IR$, find the resistance of the device.

$$R = 1 \Omega$$

Assessment Question A

What is the equation of the green line?

A) $y = -2x + 2.5$

B) $y = 2x + 2.5$

C) $y = \frac{1}{2}x + 2.5$

D) $y = -\frac{1}{2}x + 2.5$
Objective B: Parallel and Perpendicular Lines

As mentioned in the previous section, the gradient of a line indicates how steep the line is. Since parallel lines can never meet, they have the same steepness by definition! So

1. Saying that two lines are parallel is the same as saying they have equal gradient

Parallel lines cross at right-angles: clearly if one has a positive gradient, the other must have a negative gradient (otherwise they couldn’t possibly cross at right angles!) but it turns out that

2. Two lines are perpendicular if their gradients are negative reciprocals of one another
(You can prove this using Pythagoras – try it!)

In equations, if we have two lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ then

1. The two lines are parallel if $m_1 = m_2$
2. The two lines are perpendicular if $m_1 = -\frac{1}{m_2}$

(Note that the $+c$s don’t matter at all! Think: two lines can be parallel but next to each other or miles apart! Similarly, two lines can be perpendicular near the origin or really far away! That’s all the $+c$ term is doing in this context)

Worked Examples B

**Worked Example 1**

Give an example of a line that is parallel to the line given by equation $y = 3x + 4$

Since we need a line parallel to $y = 3x + 4$, we need to identify the gradient, which is the coefficient of $x$, since the equation is already in the form $y = mx + c$. The line’s gradient is 3, so any line of the form $y = 3x + c$ will work. We choose a different $c$, or else we’ll end up with the same line, so we choose $y = 3x + 1$

**Worked Example 2**

Give an example of a line that is perpendicular to the line given by equation $2y = -6x + 3$

Since we need a line perpendicular to $2y = -6x + 3$, we need to identify the gradient, which requires an extra step this time: we need to get the equation is into the form $y = mx + c$ by dividing both sides by 2.

$y = -3x + \frac{3}{2}$

so we see the line in the question has gradient -3.

The gradient of the perpendicular line is given by the negative reciprocal of the original gradient:

$m = -\frac{1}{-3} = \frac{1}{3}$

So any line of the form $y = \frac{1}{3}x + c$ will work. We choose a different $c$, or else we’ll end up with the same line, so we choose $y = \frac{1}{3}x + 2$. 


This question is more complex, but works in the same way as the previous example to start with: we see the word perpendicular, so we need to identify the gradient of the question’s line to find our gradient by the negative reciprocal (be sure to keep track of the minus signs and take care dividing by fractions!):

\[ m = -\frac{1}{-\frac{1}{2}} = -2 = 2 \]

So as before, we need a line of the form \( y = 2x + c \) but this time we don’t have free choice of the \( +c \) (if we choose \( c = 1 \) for example, we get \( y = 2x + 1 \), but we need the line to pass through \((1,10)\). Substituting this into \( y = 2x + 1 \) however leads to \( 10 = 2 \times 1 + 1 = 3 \), so \((1,10)\) isn’t on the line!)

To find \( c \), we use the fact that the line passes through \((1,10)\). This means substituting \( x = 1 \) and \( y = 10 \) into our line must work, so we can use that to find \( c \):

\[
\begin{align*}
10 &= 2 \times 1 + c \\
8 &= c
\end{align*}
\]

So the line perpendicular to \( y = -1/2 + 10 \) that also passes through \((1,10)\) is \( y = 2x + 8 \)

**Practice Questions B**

1) For each of the below, state whether the lines are parallel, perpendicular or neither

(a) \( y = -\frac{4}{5}x + 14 \) and \( y = -\frac{4}{9}x - 3 \)

parallel

(b) \( y = -\frac{3}{2}x - 1 \) and \( y = \frac{3}{2}x + 3 \)

neither

(c) \( y = x - 6 \) and \( y = -x + 4 \)

perp

(d) \( y = \frac{3}{2}x - 2 \) and \(-3x + 2y = 6 \)

parallel
2) Find lines that are parallel to the given lines and pass through the given point
   (a) (-2, -3) and \( y = -x + 2 \)
   \[
   y = -x + c \\
   -3 = 2 + c \\
   y = -x - 5
   \]

   (b) (-5, -2) and \(-3x + 2y = 6\)
   \[
   2y = 3x + 6 \\
   y = \frac{3}{2}x + c \\
   -2 = -\frac{15}{2} + c \\
   c = 5.5
   \]

   (c) (-5, 2) and \(-6x + 5y = -10\)
   \[
   y = \frac{6x - 2}{5} \\
   2 = -6 + c \\
   c = 8
   \]

3) Find lines that are parallel to the given lines and pass through the given point
   (a) (-4, 3) and \(-4x + 3y = -6\)
   \[
   y = \frac{4}{3}x - 2 \\
   3 = -\frac{16}{3} + c \\
   c = \frac{43}{3}
   \]

   (b) (3, -5) and \(4x + 9y = -9\)
   \[
   y = -\frac{4}{9}x + c \\
   -5 = -\frac{4}{3} + c \\
   c = \frac{11}{3}
   \]

   \[
   y = -\frac{4}{9}x + \frac{11}{3}
   \]
Challenge Question B

Emma plots the points A(-9,6) and B(-4,4). She claims that the line AB will be perpendicular to the line $y = 3x - 5$.

Is she correct? Explain your answer.

$$m = \frac{-9 - 4}{6 - 4} = \frac{-5}{2} \neq -\frac{1}{3}$$

(no, not correct)

Assessment Question B

Which of the following lines is not perpendicular to $y = 2x + 1$?

A) $y + \frac{1}{2}x = 6$

B) $2y = 4 - x$

C) $2x + y = 4$

D) $y = -\frac{1}{2}(7 + x)$

Objective C: Midpoints and Ratios on Straight Lines

Finding the midpoint or point from a ratio given two points on a line, the important thing is to think about each coordinate. Finding the midpoint of the $x$-coordinate and the $y$-coordinate for instance is all that needs to be done to get our answer!

Worked Examples C

**Worked Example 1**

Find the midpoint of the line segment
The coordinates are already given, which saves us a job. The important point to note is that in order to find the midpoint of (-1,2) and (7,5) we need only work out the number in the middle (midpoint!) of the \(x\)-coordinates (-1 and 7) and then the number in the middle of the \(y\)-coordinates (2 and 5).

To find the number in the middle to two numbers, we simply sum then and divide by 2:

\[
\begin{align*}
  x &= \frac{-1 + 7}{2} = \frac{6}{2} = 3 \\
  y &= \frac{2 + 5}{2} = \frac{7}{2} = 3.5
\end{align*}
\]

So the midpoint of the line segment is (3, 3.5)

---

**Worked Example 2**

Find the point C so that AC and BC are the same length

“The same length” is our clue to look for midpoints again, though this time we have to read the coordinates first. Taking care to put the \(x\)-coordinates first and \(y\)-coordinates second, we identify A(4, 2) and B(-8, -1).

Now we again find the middle of the \(x\)-coordinates and the middle of the \(y\)-coordinates:

\[
\begin{align*}
  x &= \frac{4 + -8}{2} = \frac{-4}{2} = -2 \\
  y &= \frac{2 + -1}{2} = \frac{1}{2} = 0.5
\end{align*}
\]

So (-2, 0.5) is the point C.

Note: we should always check! We can “eyeball” the middle of the line to be about here:

This gives us a good indication that the \(x\)-coordinate should be negative and the \(y\) should be positive, so we can at least be sure we got our signs right!

---

**Worked Example 3**

Find the point C such that the ratio BC : AC is 2:3

---
Now we have to use a ratio to find the point on the line, but the process is similar to that with midpoints. First we identify the coordinates A(4, 2) and B(-8, -1).

The difference appears now: we have to think about ratios. Our ratio here indicates that we have 5 parts, and from B to C we have 2 parts and from C to A we have 3 parts.

So we need to find how long BA is, split it into 5 parts and work out where C is. Instead of working with the line BA, we can work with coordinates again! We find the lengths by subtracting the coordinates:

\[ x\text{ length} = 4 - (-8) = 12 \]
\[ y\text{ length} = 2 - (-1) = 3 \]

(we can count squares and double check with the figure to makes sure this makes sense)

Now we need to share these lengths by the ratio: C is 2/5 along the line:

\[ x\text{ length to } C = \frac{2}{5} \times 12 = \frac{24}{5} = 4.8 \]
\[ y\text{ length to } C = \frac{2}{5} \times 3 = \frac{6}{5} = 1.2 \]

These are lengths however! We need to add them to our start point (coordinate B) to find the coordinates of C.

\[ x \text{ coordinate of } C = 4.8 + (-8) = -3.2 \]
\[ y \text{ coordinate of } C = 1.2 + (-1) = 0.2 \]

So C has coordinates (-3.2, 0.2), which a quick check of our sketch above seems to confirm.

**Practice Questions C**

1) M is the midpoint of the line segment AB. Find the coordinates of M.

\[ \left( 1, 1 \right) \]
2) N is the midpoint of the line segment CD. Find the coordinates of N.

\[(1, 0)\]

3) Point E has coordinates (1, 4) and point F has coordinates (34, 28). Point G lies on the line segment EF such that EG:GF = 1:2. Find the coordinates of point G.

\[EF: (33, 24) \quad \text{and} \quad G = (11, 8)\]

\[\therefore (12, 12)\]

**Challenge Question C**

Using the diagram, find a formula for the point p such that the ratio \(Bp : pA\) is n:m.
Assessment Question C

Find the midpoint of the line segment joining the points \( P_1 = (1,3) \) and \( P_2 = (6,8) \)

A) \( \left( \frac{11}{2}, \frac{7}{2} \right) \)

B) \( \left( \frac{7}{2}, \frac{11}{2} \right) \)

C) \( (-5,5) \)

D) \( (7,11) \)
Objectives this Tutorial

A) Plotting from $y = mx + c$
B) Finding the equation of a line given points/gradient
C) Solving simultaneous equations with graphs

Starter

Try to think how you might solve the following problems:

- Write down the equation of 3 lines that contain the coordinate (4,-6)
- The gradient between (5,-7) and (a,b) is 2. Find 3 possibilities for (a,b)

Objective A: Plotting from $y = mx + c$

When plotting linear graphs we end up with straight lines. The key steps in plotting a linear graphs are:

1) Make sure the line is in the correct form ($y = mx + c$)
2) Work out the range of $x$ values to plot
3) Draw a table of values
4) Substitute the $x$ values into the line equation to find the $y$ values
5) Choose appropriate scales for the axes to plot the data
6) Plot the points on the graph
7) Draw the line

Worked Examples A

Worked Example 1

Plot $y = 2x + 1$ on the axes given

Following the steps outlined above:

1) The line $y = 2x + 1$ is already in the correct form
(2) The $x$-axis runs from 0 to 5, so 0, 1, 2, 3, 4, 5 would make the most sense

(3) We draw the table of values

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4) Substitute the $x$ values into the equation $y = 2x + 1$ one by one

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$2 \times 0 + 1 = 1$</td>
<td>$2 \times 1 + 1 = 3$</td>
<td>$2 \times 2 + 1 = 5$</td>
<td>$2 \times 3 + 1 = 7$</td>
<td>$2 \times 4 + 1 = 9$</td>
<td>$2 \times 5 + 1 = 11$</td>
</tr>
</tbody>
</table>

(5) Here the axes are already pre-labelled, so there’s no need to add labels to either one

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible

(7) And finally draw the line with a ruler
This question is a little more complex – we need to careful with negatives and think a bit with part (b), but we can still follow the steps!

(1) If we write the right-hand side of the equation as \(-\frac{1}{2}x + 3\) we see the correct form

(2) The \(x\)-axis runs from -3 to 3, so -3, -2, -1, 0, 1, 2, 3 would make the most sense

(3) We draw the table of values

<table>
<thead>
<tr>
<th>(x)</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>(-\frac{1}{2} \times -3 + 3 = 4.5)</td>
<td>(-\frac{1}{2} \times -2 + 3 = 4)</td>
<td>(-\frac{1}{2} \times -1 + 3 = 3.5)</td>
<td>3</td>
<td>(-\frac{1}{2} \times 0 + 3 = 3)</td>
<td>(-\frac{1}{2} \times 1 + 3 = 2.5)</td>
<td>(-\frac{1}{2} \times 2 + 3 = 2)</td>
</tr>
</tbody>
</table>

(4) Substitute the \(x\) values into the equation \(y = -\frac{1}{2}x + 3\) one by one (being careful with negatives!

(5) Here the \(x\)-axes are already pre-labelled, so we just need to choose a scale for the \(y\)-axis. Our \(y\) values are between 1.5 and 4.5, so 0, 1, 2, 3, 4, 5, 6 is ideal (note: sometimes going up in 2s or 5s etc might make more sense!)

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible
And finally draw the line with a ruler

Now for part (b): we draw a line from \( y = 4.2 \) (as best we can given our scale) to the line we’ve drawn, then vertically downwards to see the corresponding value of \( x \): it is indeed negative, around \(-2.4\).

This question requires some thought to save some time, but otherwise more of the same. Following the steps outlined above:

1. Subtracting \( x \) from both sides gives the correct form: \( y = -x + 3 \) (writing the \( x \) first helps to identify the gradient)
2. The question asks for \( x \) values between -10 and 10, but plotting -10, -9, -8 all the way to 10 would waste a lot of time. We choose a suitable increment: -10, -5, 0, 5, 10 for example (when choosing increments, you should plot a minimum of four points to make sure you haven’t made any mistakes!)
(3) We draw the table of values

<table>
<thead>
<tr>
<th>x</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4) Substitute the x values into the equation \( y = -x + 3 \) one by one

<table>
<thead>
<tr>
<th>x</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>((-10) + 3 = 13)</td>
<td>((-5) + 3 = 8)</td>
<td>((0) + 3 = 3)</td>
<td>((-5) + 3 = -2)</td>
<td>((-10) + 3 = -7)</td>
</tr>
</tbody>
</table>

(5) Our axes are pre-labelled – note that the x- and y- scales are different. This is not a problem – the only thing to be careful of is to make sure the increment is consistent (2, 4, 6, ..., not 2, 4, 7 and with equal size measured on the graph!)

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible.

(7) And finally draw the line with a ruler.

Practice Questions A

1) a) Complete the table of values for \( y = 2x + 5 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>
b) On the grid draw the graph of \( y = 2x + 5 \) for values of \( x \) from \( x = -2 \) to \( x = 2 \).

2) On the grid draw the graph of \( y = 3x - 2 \) for values of \( x \) from \(-1 \) to \( 3 \).
3) On the grid draw the graph of 
\[ x + y = 5 \]

**Challenge Question A**

On the grid below, draw appropriate axes and plot 
\[ 3y = 2(3x - 9) \]
\[ y = 2x - 6 \]

**Assessment Question A**

Which of these lines has the equation 
\[ y = 4 - 2x \]
Objective B: Finding the Equation of a Line Given Points or Gradient

We can find all the information about a line (and therefore draw the line) if we have two pieces of information. There are two options here:

Either

4) The coordinates of a point on the line and the gradient of the line

Or

5) The coordinates of two different points on the line

Both cases work similarly, though the second takes an extra step.

If we know the gradient and a point, we can write \( y = mx + c \) and simply substitute in the coordinates of the point and rearrange to find \( c \).

If we know the coordinates of two points on the line, we must first use the formula of the gradient

\[
m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

to find the gradient, then we write \( y = mx + c \) and as before substitute in either of the given points to find \( c \).

Worked Examples B

**Worked Example 1**

Give the equation of the straight line with gradient 2 than passes through \((1, 4)\)

Since we know that the gradient is 2, we know the line must be of the form \( y = 2x + c \).

Since we also know that the line passes through \((1, 4)\), we can find \( c \) by substituting in \( x = 1 \) and \( y = 4 \) and solving for \( c \).

\[
4 = 2 \times 1 + c
\]
\[
4 = 2 + c
\]
\[
2 = c
\]

So the equation of the line is \( y = 2x + 2 \)

**Worked Example 2**

A line passes through \((1, 2)\) \((2, 8)\). What is the equation of the line?

With no gradient given, we need to find that for ourselves.

Using the formula

\[
m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

We need to identify our two points:

\((1,2)\) and \((2,8)\)

\((x_1, y_1)\) and \((x_2, y_2)\)

so the gradient is

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 2}{2 - 1} = \frac{6}{1} = 6
\]

Now we know that the line must have equation \( y = 6x + c \) and we can find \( c \) just like in the previous example: substitute in one of the points. \((1,2)\) is simpler than \((2,8)\) so let’s choose that one:

\[
2 = 6 \times 1 + c
\]
\[
\begin{align*}
2 &= 6 + c \\
-4 &= c \\
\text{so the equation of the line is } y &= 6x - 4
\end{align*}
\]

**Worked Example 3**

A line passes through \((-1, -4)\) and crosses the \(y\)-axis at \(-2\). What is the equation of the line?

This question is almost identical to example 2, but in disguise. We need to recall that if the lines crosses the \(y\)-axis at \(-2\), this coordinate is \((0, -2)\) (since every point on the \(y\)-axis has \(x\)-coordinate 0). Equipped with this, we do the same steps as before:

Using the formula

\[
m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

We need to identify our two points:

\((-1, -4)\) and \((0, -2)\)

\((x_1, y_1)\) and \((x_2, y_2)\)

so the gradient is

\[
m = \frac{-2 - (-4)}{0 - (-1)} = \frac{2}{1} = 2
\]

Now we know that the line must have equation \(y = mx + c\) and since the question directly tells us the \(y\)-intercept is \(-2\), the equation of the line is

\[y = 2x - 2\]

**Practice Questions B**

1) A line has gradient of 1.5 and passes through \((4, 5)\). What is the equation of the line?

\[y = 1.5x - 1\]

2) What is the equation of the straight line that passes through \((1, -4)\) and \((3, 2)\)?

\[y = 3x - 7\]
3) What is the equation of the straight line that passes through (-5, 2) and (7, -1)?

\[ j = -\frac{1}{4}x + \frac{3}{4} \]

4) Does the line \( y = 3x + 2 \) go through (-2, -3)? How do you know?

No! \(-3 \neq -6+2\)

**Challenge Question B**

Does the point (0.7, 1.65) lie on the line with gradient \(-\frac{1}{2}\) and \(x\)-intercept 4?

\[ y = -\frac{1}{2}x + 2 \]

So no!

**Assessment Question B**

A line has a gradient of 4 and passes through the point (1,7). What is its equation?

A) \( y = 7x + 4 \)

B) \( y = 4x + 7 \)

C) \( y = 4x + 6 \)

D) \( y = 4x + 3 \)
Objective C: Solving Simultaneous Equations with Graphs

Every point on a line is a point that satisfies the line’s equation \( y = mx + c \). If we plot two straight lines, we see clearly that there is only one crossing point (unless the lines are parallel!). This is true of any straight lines – try it!

If all the red points (forming the red line) satisfy \( y = 2x + 2 \) and all the blue points forming the blue line satisfy \( y = x - 1 \), the only point satisfying both equations is \((-3, -4)\) as it lies on both lines.

We can therefore solve simultaneous equations (pairs of equations that are true at the same time) using graphs by looking for the crossing points, though we often have to estimate the solution as it may not lie on a neat grid point.

Worked Examples C

**Worked Example 1**

Using the graph, find the solution of the pair of simultaneous equations:

\[
\begin{align*}
  y &= x + 4 \\
  y &= 6 - x
\end{align*}
\]

Since the graph lines are already drawn, this is almost entirely a comprehension exercise: recall that all the points on the red line satisfy \( y = 6 - x \) and all the points on the blue line satisfy \( y = x + 4 \) so the only point satisfying both is the point of intersection \((1, 5)\). In other words, the simultaneous equations are solved by \( x = 1 \) and \( y = 5 \).
Worked Example 2
Using the graph, find the solution of the pair of simultaneous equations
\[ y = 2x + 1 \]
\[ 2y = -2x + 8 \]
One of the lines has been drawn for you.

Here we are given one of the lines drawn but not the other. A quick look at the gradients tells us that the line already plotted must be \[ 2y = -2x + 8 \] as \[ y = 2x + 1 \] has a positive gradient unlike the already plotted line.

In order to plot \[ y = 2x + 1 \], we can construct a quick table of values (remember, we need at least four points to get a reliable line, and preferably not all next to each other) \( x = -4, 0, 4 \) and 8 should do nicely.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -4 )</th>
<th>0</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( 2 \times -4 + 1 = -7 )</td>
<td>( 2 \times 0 + 1 = 1 )</td>
<td>( 2 \times 4 + 1 = 9 )</td>
<td>( 2 \times 8 + 1 = 17 )</td>
</tr>
</tbody>
</table>

Studying our graph, it seems that these points won’t fit! Let’s try some points closer to the origin: \( x = -1, 0, 1, 2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -1 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( 2 \times -1 + 1 = -1 )</td>
<td>( 2 \times 0 + 1 = 1 )</td>
<td>( 2 \times 1 + 1 = 3 )</td>
<td>( 2 \times 2 + 1 = 5 )</td>
</tr>
</tbody>
</table>

Much better - now we plot these points on the grid and joining them with a ruled line:

And we can read off that the point of intersection is \((1,3)\) so the simultaneous equations have solution \( x = 1, y = 3 \).
Practice Questions C

1) Using the graph, write down the solutions to the following pairs of simultaneous equations

A) \( y = 2x + 4 \)  
   \( y = x \)
   
   \( x = -4 \)  
   \( y = -4 \)

B) \( y = x \)  
   \( 2x + y = 12 \)
   
   \( x = 4 \)  
   \( y = 4 \)

C) \( y = 2x + 4 \)  
   \( 2x + y = 12 \)
   
   \( x = 2 \)  
   \( y = 9 \)

2) Using the graph, write down the solutions to the following pairs of simultaneous equations

A) \( x + y = 3 \)  
   \( 2x + y = 4 \)
   
   \( x = 1 \)  
   \( y = 2 \)

B) \( x + y = 3 \)  
   \( y = x - 5 \)
   
   \( x = 4 \)  
   \( y = -1 \)

C) \( 2x + y = 4 \)  
   \( y = x - 5 \)
   
   \( x = 3 \)  
   \( y = -2 \)
3) Plot the two lines and hence solve the equations:
A) \( y = 3x - 2 \) and \( y = x - 2 \)

Point of intersection: \((0, -2)\)
so \(x = 0\) and \(y = -2\)

B) \( x + y = 5 \) and \( y = 2x - 1 \)

Point of intersection \((\frac{2}{3}, \frac{3}{3})\)
so \(x = \frac{2}{3}\) and \(y = \frac{3}{3}\)

**Challenge Question C**

Two companies are set up at the same time. The first company sells 8 TVs on their first day, and five TVs every two days from then on. The second company sells 4 TVs on their first day, and sell three TVs per day from then on.

Show on a graph of sales how many days it is until the second company's sale total exceeds the first.

1) \( \text{sales} = 8 + \frac{5}{2} \times \text{days} \)

2) \( \text{sales} = 4 + 3 \times \text{days} \)
Assessment Question C

The lines with the following equations are shown on the graph.

\[ y = 2x - 4 \quad x + y = -10 \]
\[ y = x \quad 2x + 3y = 36 \]

Write down the solutions to the following pairs of simultaneous equations:

a) \( y = 2x - 4 \) and \( y = x \)
   - \( x = 4 \)
   - \( y = 4 \)

b) \( y = 2x - 4 \) and \( 2x + 3y = 36 \)
   - \( x = 6 \)
   - \( y = 9 \)

c) \( y = x \) and \( x + y = -10 \)
   - \( x = -5 \)
   - \( y = -5 \)

d) \( y = 2x - 4 \) and \( x + y = -10 \)
   - \( x = -2 \)
   - \( y = -6 \)
Tutorial 4 – Quadratic Graphs

Objectives this Tutorial
A) Plotting a quadratic graph from an equation
B) Interpreting quadratic graphs to find roots and values
C) Approximating minima and lines of symmetry using graphs

Starter
Try to think how you might solve the following problems:
- Write down the equation of 3 quadratic curves that cross the x-axis at (6,0)
  1)  
  2)  
  3)  
- Write down the equation of 3 quadratic curves with a line of symmetry x = 5
  1)  
  2)  
  3)

Objective A: Plotting a Quadratic Graph from an Equation
When plotting a quadratic graph (an equation that contains an $x^2$ term) we end up with a parabola – the bowl shape plotted here:
The key steps in plotting a quadratic graph is the same as with linear graphs
(1) Get the equation is in the correct form ($y = ax^2 + bx + c$)
(2) Work out the range of $x$ values to plot
(3) Draw a table of values
(4) Substitute the $x$ values into the line equation to find the $y$ values (be careful with negatives!)
(5) Choose appropriate scales for the axes to plot the data
(6) Plot the points on the graph
(7) Draw the line

Worked Examples A

Worked Example 1

(a) Complete the table of values for $y = x^2 + x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) On the grid, draw the graph of $y = x^2 + x$

Following the steps outlined above:
The equation \( y = x^2 + x \) is already in the correct form (with no constant \((c)\) term)

We’ve been given the table of values already so we know our \( x \) values

We draw the table of values

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>((-3)^2 + (-3) = 6)</td>
<td>((-2)^2 + (-2) = 2)</td>
<td>((-1)^2 + (-1) = 0)</td>
<td>((0)^2 + (0) = 0)</td>
<td>((1)^2 + (1) = 2)</td>
<td>((2)^2 + (2) = 6)</td>
<td>((3)^2 + (3) = 12)</td>
</tr>
</tbody>
</table>

Substitute the \( x \) values into the equation \( y = x^2 + x \) one by one. CAUTION: when substituting \( x = -3 \) into \( x^2 + x \), a very common mistake is to write

\[-3^2 + -3 = -9 - 3 = -12\]

This is not correct! Remember, when substituting, we should include brackets to be safe!

\[(-3)^2 + (-3) = 9 - 3 = 6\]

Here the axes are already pre-labelled, so there’s no need to add labels to either one

Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible

Hint: if your points don’t look like they’re in a parabola shape, check your negative substitutions for mistakes!

And finally draw the line – here’s another key difference: because we know we’re getting a parabola, which is curved, we **don’t use a ruler**! Using a pencil will help you to get a curve that goes through the points and has the right shape

Note the bowl-shape and the symmetry of the graph!
Following the steps outlined above:

1. The equation \( y = x^2 - 2x - 4 \) is already in the correct form
2. We've been given the table of values already so we know our \( x \) values
3. We draw the table of values

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Substitute the \( x \) values into the equation \( y = x^2 - 2x - 4 \) one by one. Again, be careful with negatives!

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>((-2)^2 - 2(-2) - 4) = 4</td>
<td>((-1)^2 - 2(-1) - 4) = -1</td>
<td>((0)^2 - 2(0) - 4) = -4</td>
<td>((1)^2 - 2(1) - 4) = -5</td>
<td>((2)^2 - 2(2) - 4) = -4</td>
<td>((3)^2 - 2(3) - 4) = -1</td>
<td>((4)^2 - 2(4) - 4) = 4</td>
</tr>
</tbody>
</table>

5. Here the axes are already pre-labelled, so there’s no need to add labels to either one. Note that the scale is different on each axis!

6. Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible

Hint: if your points don’t look like they’re in a parabola shape, check your negative substitutions for mistakes!
(7) And finally draw the line – here’s another key difference: because we know we’re getting a parabola, which is curved, we **don’t use a ruler!** Using a pencil will help you to get a curve that goes through the points and has the right shape.

For part (c), we can see from the graph that the lowest value that the graph reaches is -5; this is the minimum value of y, since the graph keeps climbing in both directions (since it’s a parabola!)

(Note: this is related to the fact that we can **complete the square** – as an extension, complete the square for this function and see how we can find the minimum value of y that way):

### Worked Example 3

Plot the function $f(x) = x^2 - 4$ on the axes shown

This question looks different, but the aim is the same: plotting a quadratic graph. Highlighting the differences:

- We have $f(x)$ instead of $y$. As long as we label our ‘$y$’ axis $f(x)$ this doesn’t change anything in the question – it’s like a different label or name for the graph, but nothing else
- There is no $x$ term in the equation; again, this doesn’t matter. As long as there is any kind of $x^2$ term, we’re still plotting a quadratic, so the usual rules apply
- We aren’t provided with a table of values: then we draw our own!

Once again, following the steps outlined above:

1. The equation $f(x) = x^2 - 4$ is already in the correct form
2. The $x$-values are clearly -2, -1, 0, 1, 2, 3
3. We draw the table of values
(4) Substitute the $x$ values into the equation $y = x^2 - 4$ one by one. Again, be careful with negatives!

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$(-2)^2 - 4 = 4 - 4 = 0$</td>
<td>$(-1)^2 - 4 = 1 - 4 = -3$</td>
<td>$(0)^2 - 4 = 0 - 4 = -4$</td>
<td>$(1)^2 - 4 = 1 - 4 = -3$</td>
<td>$(2)^2 - 4 = 4 - 4 = 0$</td>
<td>$(3)^2 - 4 = 9 - 4 = 5$</td>
</tr>
</tbody>
</table>

(5) Here the axes are already pre-labelled, so there's no need to add labels to either one. Note that the scale is different on each axis!

(6) Using a pencil (easier to correct any unfortunate mistakes than pen!) plot the coordinates as accurately as possible. Hint: if your points don’t look like they’re in a parabola shape, check your negative substitutions for mistakes!

(7) And finally draw the line – here’s another key difference: because we know we’re getting a parabola, which is curved, we **don’t use a ruler**! Using a pencil will help you to get a curve that goes through the points and has the right shape.
Practice Questions A

(1) (a) For each equation, complete the table of values
(b) Graph the function on the axes shown, labelling each one.

\[ y = x^2 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

\[ y = x^2 + 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>16</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

\[ y = x^2 - 3 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>6</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>-2</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

\[ y = x^2 + 2x \]

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>8</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

\[ y = x^2 - 2x - 3 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>12</td>
<td>5</td>
<td>0</td>
<td>-3</td>
<td>-4</td>
<td>-3</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ y = 2x^2 - 4x + 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>7</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-7</td>
</tr>
</tbody>
</table>
2) On the grid below, draw the graph of \( y = x^2 - 3x - 1 \) and use your graph to estimate the minimum value of \( y \).

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>y</strong></td>
<td>9</td>
<td>3</td>
<td>-1</td>
<td>-3</td>
<td>-3</td>
<td>-1</td>
<td>3</td>
</tr>
</tbody>
</table>

(3) On the axes below, graph the function \( f(x) = x^2 - x + 3 \).

<table>
<thead>
<tr>
<th></th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>y</strong></td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>15</td>
<td>23</td>
<td>33</td>
</tr>
</tbody>
</table>
Challenge Question A

The height of a cannon ball over time is given by the equation \( h(t) = -3t^2 + 18t \).

(a) Graph the height of the cannonball over time

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(t) )</td>
<td>0</td>
<td>15</td>
<td>24</td>
<td>27</td>
<td>24</td>
<td>15</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) Find the maximum height of the cannonball

27m

(c) How many seconds after firing does the cannonball achieve maximum height?

3 seconds

Assessment Question A

Which graph shows the plot of the equation \( y = x^2 - 3x + 2 \)?
Objective B: Interpreting Quadratic Graphs to Find Roots and Values

Finding roots is one of the main reasons to plot quadratic equations. A root of a quadratic $ax^2 + bx + c$ is simply a value of $x$ which solves the equation $ax^2 + bx + c = 0$. We already know that we can solve this equation by

- Factorising into double brackets
- Completing the square
- Using the quadratic formula

However, when given a quadratic graph, we don’t need to do any of these.

Recall what drawing a graph actually means: every point on the curve is a solution to the equation $y = ax^2 + bx + c$. If we want the values of $x$ such that $ax^2 + bx + c = 0$, then it should be clear that we need to look at the points where $y = 0$ (i.e. where the graph crosses the line $x = 0$)! The values of $x$ at these crossing points satisfy $y = ax^2 + bx + c$ since they’re on the curve, and the value of $y$ here is 0, so we have our roots by simple observation!

[note: this is actually a sort of special case of a simultaneous equation! We have one line given by our equation $y = ax^2 + bx + c$ and another line given by $y = 0$ and we look for the intersections!]

When finding values rather than roots, the same logic applies, but a question might ask for us to find the value of $x$ such that $x^2 + 2x + 1 = 3$, so we find the intersection of the graph $x^2 + 2x + 1$ with $y = 3$.

**Worked Examples B**

**Worked Example 1**

The figure shows the graph $y = x^2 - 2x - 4$. Using the graph, estimate the roots of the equation

Remember, the roots of the equation are the $x$-coordinates of the points where the graph intersects the line $y = 0$ (i.e. the $x$-axis)

We can see here that the roots here are $x = -1.25$ and $x = 3.25$

[note: there are some quadratics that have only one root or even no roots (at least that we can see...) can you think of any such a quadratics? Try drawing one, then thinking of an equation!]
Instead of the roots, which means we need to look for the intersections between the graph and the line \( y = 0 \), here we need to find the intersection between the graph and the line \( y = 1.6 \). This looks like an awkward number, but if we look at the scale on our graph it's not too bad!

The scale on the \( y \)-axis is 5 little ticks = 2 so 1 little tick = \( 2 \div 5 = 0.4 \). So 1.6 is the line just below 2. Drawing that line and tracing the \( x \)-values:

Here we can see that \( x = -1.8 \) and \( x = 0.9 \) are our solutions.

Practice Questions B

1) Using the graphs whose equations are indicated in red, find estimates of the solutions to the following equations

(a) \( x^2 - 2x + 5 = 0 \)

\( x = \frac{1, 6}{1} \)

(b) \( x^2 + x - 1 = 0 \)

\( x = -1.7, 0.6 \)

(c) \( x^2 - 2x - 13 = 0 \)

\( x = 3, 5 \)
2) Using the graphs whose equations are indicated in red, find estimates of the solutions to the following equations

(a) Solve \( x^2 + 2x - 1 = 2 \)
\[ x = -3, 1 \]

(b) Solve \( x^2 - x + 1 = 7 \)
\[ x = -2, 3 \]

(c) Solve \( 2x^2 - 5x - 1 = 0 \)
\[ x = 0, 2.5 \]

3) The graph of \( y = f(x) \) is drawn on the grid.
   a) Write down the roots of \( f(x) = 0 \)
      \[ -2.7 \text{ & } 0.7 \]
   b) Use the graph to estimate the roots of \( f(x) = -1 \)
      \[ -2.4 \text{ & } 0.4 \]
   c) Write down the coordinates of the turning point of the graph
      \[ (-1, -3) \]

Challenge Question B

The grid below shows the graph of \( y = 2x^2 - 4x + 1 \)

The graph of \( 2x^2 - 4x + 1 = k \) has exactly one solution.

Use the graph to find the value of \( k \).
\[ k = -1 \]

Assessment Question B

This is the graph of \( y = x^2 - 3x - 2 \).

Using the graph, which of the following answers provides the best approximation to the solution of \( x^2 - 3x - 2 = 3 \)?

A) \( x = -1.2, x = 4.2 \)

B) \( x = -0.5, x = 3.5 \)

C) \( x = -2 \)

D) \( x = 1.2, x = -4.2 \)
Objective C: Approximating Minima and Symmetry Using Graphs

As we can see from all the quadratics we’ve plotted so far, we can see that each and every quadratic when plotted gives a parabola.

Every parabola plotted from the form $y = x^2 + bx + c$ is symmetrical along a vertical line of symmetry (in this figure $x = 1$).

Every parabola plotted from the form $y = x^2 + bx + c$ has a minimum value (in this figure $y = -5$).

[This is again related to the fact that we can complete the square given a quadratic equation]

We can see these on a plot much easier than completing the square however!

Worked Examples C

**Worked Example 1**

The figure shows the graph $y = x^2 + x$.

Using the graph, estimate:
(a) The minimum value of $y$
(b) The line of symmetry of $y$

The minimum value of $y$ here is tricky to identify but appears to be a little above the first minor gridline below zero. Each minor gridline in the $y$-axis is $2 \div 5 = 0.4$, so the minimum value of $y$ is around $-0.4$.

For the line of symmetry, it is often best to identify to points with the same $y$-value: in this instance, I can see that the graph cuts the $x$-axis at 0 and -1. I know that the graph is symmetrical, so my line of symmetry is half way between those points, at $x = -0.5$
This question uses some different language, but for the same maths. For the first part we need to find the line of symmetry, and for the second we need to identify the turning point (the point at which the graph turns – the minimum!). The use of the word “approximate” is used here because we have to draw lines and use our best judgement for the \(x\)- and \(y\)-values.

Again, for identifying the line of symmetry we find two points on the curve with the same \(y\)-value. We can choose the \(x\)-intercepts, but we can also see that the curve has the \(y\)-value of -4 at both \(x = 0\) and \(x = 2\) so half-way between those two gives \(x = 1\) as the line of symmetry.

For the turning point, we need to identify the coordinates of the point highlighted in gold; the minimum of the graph. The \(y\)-coordinate we can estimate on the graph to be around \(y = -3\). We already found the \(x\)-coordinate in finding the axis of symmetry: \(x = 1\)

Therefore we can use the graph to approximate the turning point to be (1, -3)

**Practice Questions C**

(1) The following figure shows five quadratics. By factorising the following equations, identify the curve and find
   a. The minimum value of this graph
   b. The line of symmetry of this graph

\[
\begin{align*}
\text{blue:} & \quad y = x^2 - 8x + 7 \quad \text{minimum:} \ (4, -9) \quad \text{line of symmetry:} \ x = 4 \\
\text{green:} & \quad y = x^2 - 8x + 15 \quad \text{minimum:} \ (4, -1) \quad \text{line of symmetry:} \ x = 4 \\
\text{red:} & \quad y = x^2 - 5x - 14 \quad \text{minimum:} \ (2.5, -20) \quad \text{line of symmetry:} \ x = 2.5 \\
\text{purple:} & \quad y = x^2 + 12x + 35 \quad \text{minimum:} \ (-6, -1) \quad \text{line of symmetry:} \ x = -6 \\
\text{grey:} & \quad y = x^2 + 2x - 8 \quad \text{minimum:} \ (-1, -9) \quad \text{line of symmetry:} \ x = -1
\end{align*}
\]
(2) Looking at your answers from the above, can you find a way to predict the line of symmetry for a function? What about the minimum?

\[ \text{With } x^2 + px + q \quad \text{the line of symmetry is } x = -\frac{p}{2} \]

\[ \text{Minimum } = \left( -\frac{p}{2}, \frac{p^2}{4} - \frac{p^2}{2} + q \right) \]

\[ = \left( -\frac{p}{2}, q - \frac{p^2}{4} \right) \]

(3) Find the coordinates of the turning points for the function \( y = x^2 - 5x + 5 \) (can be done using the graph above!)

\[ x^2 - 5x + 5 = 0 \]

\[ x^2 - 5x - 14 = -19 \]

\[ (\text{red}) \]

so \( x = 1.5 \) & \( 3.5 \)
Challenge Question C

(a) A quadratic has the form \( x^2 + bx + c \). What is the line of symmetry of the quadratic? What is its minimum value?

See q2.

(b) We need to use different terminology with quadratics of the form \(-x^2 + bx + c\). Why does the term minimum no longer make any sense? What word should we use instead?

graph looks like \( \bigvee \) so maximum not minimum

Assessment Question C

A parabola has a turning point at (-3,2). Where is the axis of symmetry?

A) \( y = -2 \)

B) \( x = 3 \)

C) \( x = -3 \)

D) \( y = 2 \)
Tutorial 5 – Advanced Graphs

Objectives this Tutorial
A) Solving quadratic simultaneous equations using a graph
B) Recognising advanced graphs
C) Plotting advanced graphs

Starter
Try to think how you might solve the following problems:

• Write down the equation of 3 graphs that cross the x-axis at (3,0)
  1) 
  2) 
  3) 

• Write down the coordinates of 3 points on the graph of \( y = 4^x \) that all have negative x-values
  1) 
  2) 
  3) 

Objective A: Solving Quadratic Simultaneous Equations Using a Graph

Just as in tutorial 3, solving simultaneous equations where one of the equations is a quadratic equation can be done using graphs simply by looking for the points where the two graphs meet! This is because each graph is the set of points that solve the equation of that graph, so in order to solve both equations the solution must lie on both graphs (where the lines meet!)

Questions in this area often provide one graph and will ask you to draw “an appropriate line” to find the solutions. The first worked example below has the full breakdown, whereas the second requires us to put the pieces together ourselves.
Worked Examples A

Worked Example 1

Example
a) Draw the graph of \( y = x^2 - 9 \)
b) Explain why your graph shows the solutions
to \( x^2 - 9 = 0 \) are -3 and 3
c) By drawing a suitable line find approximate solutions to \( x^2 - 9 = 6 \)
d) Draw the line \( y = 4x - 12 \)
e) Use your answer to part d) to find the solutions to \( x^2 - 9 = 4x - 12 \)

This question combines a lot of the work from our previous tutorials. For part a) we fill in the table (being careful not to muddle up our minus signs!)

<table>
<thead>
<tr>
<th></th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>(−4)^2 − 9</td>
<td>=7</td>
<td>(−3)^2 − 9</td>
<td>=0</td>
<td>(−2)^2 − 9</td>
<td>=5</td>
<td>(−1)^2 − 9</td>
<td>=8</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1)^2 − 9</td>
<td>=8</td>
<td>(2)^2 − 9</td>
<td>=5</td>
<td>(3)^2 − 9</td>
</tr>
</tbody>
</table>

Then we plot the points on our graph and remember: quadratics are curves, not straight lines!
For part b) recall that the points on the graph are the points where \( y = x^2 - 9 \) so the points that lie on the line \( y = 0 \) (the x-axis) are our solutions: from the graph we can see that we have 3 and -3 as the question suggests!

The same logic works for part c), but this time we need to draw a line. In order to find the solutions to \( x^2 - 9 = 6 \), we need to draw the line \( y = 6 \) and find the crossing points

Here we can see that the crossing points are at around \( x = 3.75 \) and \( x = -3.75 \) (NOTE: when dealing with quadratics, we will often have TWO solutions to questions, because of the shape of the graph!)

For part d) we draw a table of values and plot the straight line \( y = 4x - 12 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>4x -4 - 12 = -28</td>
<td>4x -3 - 12 = -24</td>
<td>4x -2 - 12 = -20</td>
<td>4x -1 - 12 = -16</td>
<td>4x 0 - 12 = -12</td>
<td>4x 1 - 12 = -8</td>
<td>4x 2 - 12 = -4</td>
<td>4x 3 - 12 = 0</td>
<td>4x 4 - 12 = 4</td>
</tr>
</tbody>
</table>
And finally we can answer part e): the solutions to the equation $x^2 - 9 = 4x - 12$ are the points where the graphs of $y = x^2 - 9$ and $y = 4x - 12$ cross, which we can see here happens at $x = 1$ and $x = 3$.

This question has fewer parts, but it does give us the quadratic graph already which saves us some time!

Part (a) is asking us to solve the equation $x^2 + 2x + 3 = x + 5$: we already have the graph of the left-hand side plotted, so all we need to do is plot the graph $y = x + 5$ and find the crossing points. We don’t need to choose all the $x$-values, but we need a spread:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-3</th>
<th>0</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-6 + 5 = -1</td>
<td>-3 + 5 = 2</td>
<td>0 + 5 = 5</td>
<td>3 + 5 = 8</td>
<td>6 + 5 = 11</td>
</tr>
</tbody>
</table>
Looking at the crossing points, we see the solutions are $x = -2$ and $x = 1$.

Now for the tricky part, part (b): are we supposed to plot another quadratic?! These are tough GCSE questions, but are often a lot more straightforward than they look. One clue here is that this is a part (b), not an entirely new question, so probably involves the graphs we’ve already drawn. Another is the ‘hence’ which means we should be able to use part (a) to solve part (b).

Here’s the trick:

\[
\begin{align*}
+1 - x & \quad x^2 + 3x + 2 = 2x + 4 \\
+1 - x & \quad x^2 + 2x + 3 = x + 5
\end{align*}
\]

So the solutions to part (b) are exactly the same as those to part (a)! A trick on questions like this is to always try to rearrange your new equation so that one side is the quadratic you’ve already drawn. Sometimes the other equation might be different so you might have to draw another straight line, but that’s much easier than trying to get another quadratic!

**Practice Questions A**

1) Plotted is the graph $y = x^2 + x - 1$. By plotting the graph $y = x + 3$, find the solutions to the equation $x^2 + x - 1 = x + 3$
2) By plotting the appropriate lines, show that the solutions of $x^2 - x - 2 = x + 2$ lie in the regions $-2 \leq x \leq -1$ and $3 \leq x \leq 4$. 

Crossing points between $x = -1$ & $x = -2$ and $x = 3 \& x = 4$. 
3) The graph shows the line $y = x^2 - 3x$. By plotting an appropriate line, estimate the negative solution to the equation $x^2 - 4.5x + 1 = 0$.

$$x^2 - 3x = 1.5x + 1$$

Negative crossing point $\Rightarrow x = -0.2$
Challenge Question A

The graph shows the plot of \( y = x^3 - 3x^2 + x + 1 \). By plotting an appropriate line, find the solutions to the equation \( y = x^3 - 3x^2 + x + 1 \):

\[
x^3 - 3x^2 + x + 1 = 0
\]

Since \( y = x^3 - 3x^2 + x + 1 \), the crossing points are:

\[ x = -0.55, 0.7, 2.85 \]

Assessment Question A

Here is the graph of \( y = x^2 + 3x - 1 \)

What line would you need to draw to find an approximate solution to \( x^2 + 4x - 4 = 0 \)?

A) \( y = 0 \)  
B) \( y = 3 - x \)  
C) \( y = x - 3 \)  
D) \( y = -3 \)
**Objective B: Recognising Advanced Graphs**

There are of course more graphs than just linear and quadratic (as the challenge question just now revealed!). Here are some more graphs we should be able to recognise.

<table>
<thead>
<tr>
<th>Graph type</th>
<th>Important features</th>
<th>Sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear graphs</td>
<td>$y = mx + c$&lt;br&gt;Gradient, intercept&lt;br&gt; Straight line (ruler!)&lt;br&gt; $m$ positive (1) or negative (2)</td>
<td><img src="1" alt="Sketch" /> <img src="2" alt="Sketch" /></td>
</tr>
<tr>
<td>Quadratic graphs</td>
<td>$y = ax^2 + bx + c$&lt;br&gt; $a$ positive (1) or negative (2)?&lt;br&gt; Parabola shape - curve</td>
<td><img src="1" alt="Sketch" /> <img src="2" alt="Sketch" /></td>
</tr>
<tr>
<td>Cubic graphs</td>
<td>$y = ax^3 + bx^2 + cx + d$&lt;br&gt; $a$ positive (1) or negative (2)?&lt;br&gt; A curve up and down</td>
<td><img src="1" alt="Sketch" /> <img src="2" alt="Sketch" /></td>
</tr>
<tr>
<td>Reciprocal graphs</td>
<td>$y = \frac{1}{x}$&lt;br&gt; Not connected!</td>
<td><img src="1" alt="Sketch" /> <img src="2" alt="Sketch" /></td>
</tr>
<tr>
<td>Exponential graphs</td>
<td>$y = ax^t$&lt;br&gt; Always passes through $(0,1)$</td>
<td><img src="1" alt="Sketch" /> <img src="2" alt="Sketch" /></td>
</tr>
<tr>
<td>(*)Circle graphs</td>
<td>$x^2 + y^2 = r^2$&lt;br&gt; $r$ is the radius of the circle centred at $(0,0)$</td>
<td><img src="1" alt="Sketch" /> <img src="2" alt="Sketch" /></td>
</tr>
<tr>
<td>(*)Square root graphs</td>
<td>$y = \sqrt{x}$&lt;br&gt; Half a parabola on its side</td>
<td><img src="1" alt="Sketch" /> <img src="2" alt="Sketch" /></td>
</tr>
</tbody>
</table>

* these graphs are used in other areas of maths but we won’t go into depth with them in this book.
First of all we have to distinguish between our two straight lines. The one with the positive gradient is the green one because it traces a line from bottom left to top right. (If in doubt, pick a simple line like $y = x + 1$ and do a quick table of values to see what a positive gradient graph would look like!)

Then we have the purple and the red for cubic and exponential. The cubic graph always has an up and a down curve (even if there’s a curve in the middle, like here!) so red is the cubic curve.

This leaves purple to be the exponential, which looks good because exponential graphs are of the form e.g. $2^x$. Importantly this can’t be negative for any value of $x$ (try it and see!) so the purple line makes sense.

This question is nice and straightforward, but can confuse us with the coefficients in the equations.

More important than the coefficients are the overall shapes! For example equation 1 is clearly a straight line graph – there is only an $x$ term, no powers etc – only a gradient. The only straight line option for us is graph b).

Similarly graph a) is the only quadratic graph (the give-away parabola bowl shape!) and the only option that’s quadratic is equation 2.

The other two: one cubic ($x^3$) and one reciprocal ($\frac{1}{x}$) graph. A good tip here is to think that $x$, $x^2$ and $x^3$ all follow a pattern, where $\frac{1}{x}$ is the odd one out – the graphs also have an odd one out – graph c is broken into two pieces! So equation 3 is graph d and equation 4 is graph c.
Practice Questions B

(1) Match the equation to the graph

<table>
<thead>
<tr>
<th>Equation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 3x + 2$</td>
<td></td>
</tr>
<tr>
<td>$y = 2x + 3$</td>
<td></td>
</tr>
<tr>
<td>$y = x^2 + 3$</td>
<td></td>
</tr>
<tr>
<td>$y = -x^2 + 3$</td>
<td></td>
</tr>
<tr>
<td>$y = x^3 + 3$</td>
<td></td>
</tr>
<tr>
<td>$y = -x^3 + 3$</td>
<td></td>
</tr>
<tr>
<td>$y = x^3 - x^2 + 3$</td>
<td></td>
</tr>
<tr>
<td>$y = x^2 + 2x + 3$</td>
<td></td>
</tr>
<tr>
<td>$y = \frac{1}{x}$</td>
<td></td>
</tr>
<tr>
<td>$y = 2^x$</td>
<td></td>
</tr>
</tbody>
</table>
(2) Match each of the seven graphs to one of the seven equations below. You must justify your answer!

A) \( y = x^2 + 2 \)
Why? Downward parabola, positive y-intercept

B) \( y = -2x + 3 \)
Why? Linear, negative gradient, positive intercept

C) \( y = \frac{1}{x} \)
Why? Only reciprocal graph

D) \( y = 2^x \)
Why? Only exponential graph

E) \( y = x^3 + 2 \)
Why? Positive cubic graph - see y-scale for confusion over intercept

F) \( y = 3x + 5 \)
Why? Linear, positive gradient, positive y-intercept

G) \( y = -x^2 + 5 \)
Why? Upward parabola, positive y-intercept

<table>
<thead>
<tr>
<th>Equation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 3x + 5 )</td>
<td>![Graph of y = 3x + 5]</td>
</tr>
<tr>
<td>( y = x^2 + 2 )</td>
<td>![Graph of y = x^2 + 2]</td>
</tr>
<tr>
<td>( y = \frac{1}{x} )</td>
<td>![Graph of y = 1/x]</td>
</tr>
<tr>
<td>( y = -2x + 3 )</td>
<td>![Graph of y = -2x + 3]</td>
</tr>
<tr>
<td>( y = x^3 + 2 )</td>
<td>![Graph of y = x^3 + 2]</td>
</tr>
<tr>
<td>( y = -x^2 + 5 )</td>
<td>![Graph of y = -x^2 + 5]</td>
</tr>
<tr>
<td>( y = 2^x )</td>
<td>![Graph of y = 2^x]</td>
</tr>
</tbody>
</table>
Challenge Question B

Match the equation to the graph

Assessment Question B

What is the equation of the curve shown plotted here?

A) $y = 5^x$  
B) $y = x^2 + 1$  
C) $y = \frac{5}{x}$  
D) $y = 5x^3 + 1$
Objective C: Plotting Advanced Graphs

Plotting advanced graphs may look intimidating but is no more difficult than plotting any other kind of graph! The important thing to have in mind is the shape you’re expecting out the end so you can check that your plot makes sense when you’re done.

As before, plotting a graph is easiest with a table of values, then we can follow the same instructions as before:

1. Make sure the equation is in the correct form ($y = f(x)$, in other words all the $x$s are on one side)
2. Work out the range of $x$ values to plot
3. Draw a table of values
4. Substitute the $x$ values into the line equation to find the $y$ values (be careful with negatives!)
5. Choose appropriate scales for the axes to plot the data
6. Plot the points on the graph
7. Draw the line – remember if the line is not a linear graph it should be a curve

Worked Examples C

**Worked Example 1**

Draw the graph of

$$y = x^3 + 4x^2 + x + 1$$

For values of $x$ between -4 and 2.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$(-4)^3 + 4(-4)^2 + (-4)+1$</td>
<td>$(-3)^3 + 4(-3)^2 + (-3)+1$</td>
<td>$(-2)^3 + 4(-2)^2 + (-2)+1$</td>
<td>$(-1)^3 + 4(-1)^2 + (-1)+1$</td>
<td>$(0)^3 + 4(0)^2 + (0)+1$</td>
<td>$(1)^3 + 4(1)^2 + (1)+1$</td>
<td>$(2)^3 + 4(2)^2 + (2)+1$</td>
</tr>
<tr>
<td></td>
<td>$-3$</td>
<td>$7$</td>
<td>$7$</td>
<td>$3$</td>
<td>$1$</td>
<td>$7$</td>
<td>$27$</td>
</tr>
</tbody>
</table>

Then we plot our points:

And when connecting the points, we have to think about the overall shape we should be aiming for: a nice smooth curve with a wiggle in the middle.

(again, if in doubt, say ‘what happens between -3 and -2?’ – substitute in $x = -2.5$ and see!)
**Worked Example 2**

Draw the graph of

\[ y = \frac{4}{x} \]

Between \( x = -5 \) and 5.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-0.8</td>
<td>-1</td>
<td>-1.33..</td>
<td>-2</td>
<td>-4</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ \frac{4}{-5} = -0.8 \]
\[ \frac{4}{-4} = -1 \]
\[ \frac{4}{-3} = -1.33.. \]
\[ \frac{4}{-2} = -2 \]
\[ \frac{4}{-1} = -4 \]
\[ \frac{4}{0} \]
\[ \frac{4}{1} = 4 \]
\[ \frac{4}{2} = 2 \]
\[ \frac{4}{3} = 1.33.. \]
\[ \frac{4}{4} = 1 \]
\[ \frac{4}{5} = 0.8 \]

**A big problem here:** there is no value for \( x = 0 \)! And we know that there can’t be since we can’t divide by zero (‘how many zeroes go into 4’ doesn’t even make sense as a question!) this is a good indicator and confirmation of our reciprocal graph where we have the two different parts, but let's plot and see:

Again, with the points plotted the image we should be thinking of is the two slopes as seen in the previous section, but if we want to confirm that we can put in some more points – e.g. \( x = \frac{1}{2}, x = -\frac{1}{2} \) might help!

And again we draw the curve (not straight lines).
Practice Questions C

(1) Plot the graph of $y = x^3 + 2x^2 - 3x + 1$ for values of $x$ between -4 and 2.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-19</td>
<td>1</td>
<td>7</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>11</td>
</tr>
</tbody>
</table>

(2) Draw the graph of $y = 4^x$ for $x$ values between -2 and 2.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.25</td>
<td>0.25</td>
<td>1</td>
<td>4</td>
<td>16</td>
</tr>
</tbody>
</table>

(3) Plot the graph $y = -\frac{2}{x} + 1$ for $x$ values between -5 and 5.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>05</td>
<td>03</td>
<td>02</td>
<td>01</td>
</tr>
</tbody>
</table>
Challenge Question C

At day \( t = 0 \) a radioactive material has mass 10kg. The material decays and loses mass radioactively by the equation \( m = 10e^{-0.5t} \) where \( e \) is Euler’s number (see your calculator \( e^x \)). By plotting this function, find out how much of the mass is left after 4 days.

Assessment Question C

This is a table of values for \( y = x^3 - 3x + 1 \)

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-2</td>
</tr>
</tbody>
</table>

What value should replace the star?

A) -11

B) 1

C) -1

D) -13
UP Y2 Baseline

SF
1.

(a) N is the point (-3, 0). On the grid, mark the point N with a cross, and label it N.

(b) Find the coordinates of the midpoint of the line NB.

(c) Point M is lies along the line NA such that NM:MA is 1:2. Find the coordinates of M.
2.

(a) On the grid above, draw the graph of \( y = 3 - 2x \) from \( x = -2 \) to \( x = 4 \)  

(b) Using your graph, find the value of \( x \) when \( y = -1.5 \)  
\[ x = \ldots \ldots \ldots \]  

(c) \( f(x) = -1 + 3x \) is a function of \( x \). On the grid above, draw \( y = f(x) \)  

(d) Using your graph, estimate the coordinates of the point of intersection between the lines drawn in parts (a) and (c)  
\[ (\ldots\ldots, \ldots\ldots) \]
3. (a) For the graph above, write the equation of the line, labelling clearly the key features.

(b) On the above axes, draw the graph of the line that passes through (-1,1) with a gradient of -2.

4. (a) Write down the gradient and y-intercept of the line $3y = 2(3 - 2x)$

   Gradient = ................. y-intercept = .................

(b) A line has equation $y = -\frac{2}{3}x + 3$. Give

(i) The equation of a line parallel to $L_1$

(ii) The equation of a line perpendicular to $L_1$
5.

(a) On the axes above, draw the graph of \( y = x^2 + 3x - 3 \)

(b) Use your graph to estimate

(i) The roots of \( x^2 + 3x - 3 \)

(ii) The minimum value of \( x^2 + 3x - 3 \)

(iii) The line of symmetry of \( x^2 + 3x - 3 \)
On the axes above, are plotted three graphs:

(blue): \( y = x - 1 \) and \( 3y + 2x = 12 \)
(red): \( y = x^2 + x - 6 \)

Use the above diagram to estimate the solutions to the pairs of simultaneous equations:

(a) \( y = x - 1 \) and \( 3y + 2x = 12 \)

....................................................................................................................... \( 2 \)

(b) \( y = x - 1 \) and \( y = x^2 + x - 6 \)

....................................................................................................................... \( 2 \)
7.

(a) On the axes above, plot the graph of \( y = x^3 - 3x^2 - 6x + 8 \) \( \quad \) (3)

(b) Using your graph, draw an appropriate line to estimate the solutions to the following equation
\[ x^3 - 3x^2 - 6x + 8 = -4 \] \( \quad \) (3)
8. For each of the following graphs, use one of the following descriptions

- Linear
- Exponential
- Cubic
- Square root
- Reciprocal
- Quadratic

(i) ![Graph i](image1.png)

(ii) ![Graph ii](image2.png)

(iii) ![Graph iii](image3.png)

(iv) ![Graph iv](image4.png)

(v) ![Graph v](image5.png)
UP Y2 Baseline
1.

(a) N is the point (-3, 0). On the grid, mark the point N with a cross, and label it N.

1 mark for correctly identifying the point above (1)

(b) Find the coordinates of the midpoint of the line NB

(0.5, 0) (1)

(c) Point M is lies along the line NA such that NM:MA is 1:2. Find the coordinates of M.

(-1, 1.5) (1)
2.  

(a) On the grid above, draw the graph of \( y = 3 - 2x \) from \( x = -2 \) to \( x = 4 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>-3</td>
<td>-5</td>
</tr>
</tbody>
</table>

1 mark: two mistakes in calculating values OR plotting points  
2 marks: one mistake in calculating values OR plotting points  
3 marks: line correctly plotted as above

(b) Using your graph, find the value of \( x \) when \( y = -1.5 \)

\[ x = \boxed{2.25} \]  
(Allow values between 2.1 and 2.4)

(c) \( f(x) = -1 + 3x \) is a function of \( x \). On the grid above, draw \( y = f(x) \)

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-7</td>
<td>-4</td>
<td>-1</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>11</td>
</tr>
</tbody>
</table>

1 mark: two mistakes in calculating values OR plotting points  
2 marks: one mistake in calculating values OR plotting points  
3 marks: line correctly plotted as above

(d) Using your graph, estimate the coordinates of the point of intersection between the lines drawn in parts (a) and (c)

\[ (\boxed{0.8, 1.25}) \]  
Allow \( x \) coordinate between 0.7 and 0.9  
Allow \( y \) coordinate between 1.1 and 1.4
3. (a) For the graph above, write the equation of the line, labelling clearly the key features
   \[ y = \frac{1}{3}x + 4 : 1 \text{ mark for the equation, 1 each for labelling the gradient and } y\text{-intercept} \]

(b) On the above axes, draw the graph of the line that passes through (-1,1) with a gradient of -2
   \[ 1 \text{ mark for finding the equation } y = -2x - 1 \]
   \[ 1 \text{ mark for plotting the line above with 1 mistake in table of values/plotting} \]
   \[ 2 \text{ marks for no mistakes in plot} \]

4. (a) Write down the gradient and \( y\)-intercept of the line \( 3y = 2(3 - 2x) \)

   \[ \text{Gradient } = \ldots... -\frac{4}{3} \ldots... \]
   \[ \text{y-intercept } = \ldots... \frac{2}{3} \ldots... \]
   \[ (1 \text{ mark each. Allow decimals}) \]

(b) A line has equation \( y = -\frac{2}{3}x + 3 \). Give

   (i) The equation of a line parallel to \( L_1 \)

   \[ y = -\frac{2}{3}x + [\text{any number}] \]
   \[ (1) \]

   (ii) The equation of a line perpendicular to \( L_1 \) (Allow decimals)

   \[ y = \frac{3}{2}x + [\text{any number}] \]
   \[ (1) \]
5. (a) On the axes above, draw the graph of \( y = x^2 + 3x - 3 \)

(b) Use your graph to estimate

(i) The roots of \( x^2 + 3x - 3 \)

\[ 0.8 \text{ and } -3.8 \] (1 mark each, allow \( \pm 0.1 \) on each)

(ii) The minimum value of \( x^2 + 3x - 3 \)

\[ -5.5 \] (allow \( \pm 0.2 \))

(ii) The line of symmetry of \( x^2 + 3x - 3 \)

\[ x = -1.5 \]
On the axes above, are plotted three graphs:

(blue): \( y = x - 1 \) and \( 3y + 2x = 12 \)
(red): \( y = x^2 + x - 6 \)

Use the above diagram to estimate the solutions to the pairs of simultaneous equations:

(a) \( y = x - 1 \) and \( 3y + 2x = 12 \)

\[ \begin{align*}
(3,2) & \quad \text{so } x = 3, y = 2 \quad (1) \text{ mark for just (3,2)} \\
\end{align*} \]

(b) \( y = x - 1 \) and \( y = x^2 + x - 6 \)

\[ \begin{align*}
(2.2, 1.1) & \quad \text{so } x = 2.2, y = 1.1 \\
(-2.2, 3.2) & \quad \text{so } x = -2.2, y = -3.2 \\
\end{align*} \]

(1) for each correct pair
Allow \( \pm 0.1 \) for each
7.

(a) On the axes above, plot the graph of \( y = x^3 - 3x^2 - 6x + 8 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-28</td>
<td>0</td>
<td>10</td>
<td>8</td>
<td>0</td>
<td>-8</td>
<td>-10</td>
<td>0</td>
<td>53</td>
</tr>
</tbody>
</table>

1 mark for correct table of values, 2 marks for correct graph
(nb: mark for shape and points plotted)

(b) Using your graph, draw an appropriate line to estimate the solutions to the following equation
\[ x^3 - 3x^2 - 6x + 8 = -4 \]

\[ x = -2.4, \quad x = 1.5 \quad \text{and} \quad x = 3.6 \]

1 mark each, allow \( \pm 0.2 \)
8. For each of the following graphs, use one of the following descriptions:

- Linear
- Exponential
- Cubic
- Square root
- Reciprocal
- Quadratic

(i) Exponential

(ii) Linear

(iii) Reciprocal

(iv) Quadratic

(v) Cubic
Uni Pathways Year 2

Final Test

Name: ______________________
1.

(a) N is the point (-3, -5). On the grid, mark the point N with a cross, and label it N.

(b) Find the coordinates of the midpoint of the line NA

\((\ldots, \ldots)\)

(1)

(c) Point M is lies along the line NB such that NM:MB is 1:2. Find the coordinates of M.

\((\ldots, \ldots)\)

(1)
2.

(a) On the grid above, draw the graph of \( y = 1 - 2x \) from \( x = -3 \) to \( x = 3 \)  

(b) Using your graph, find the value of \( x \) when \( y = -1.5 \)  

(c) \( f(x) = -2 + x \) is a function of \( x \). On the grid above, draw \( y = f(x) \)  

(d) Using your graph, estimate the coordinates of the point of intersection between the lines drawn in parts (a) and (c)  

\( (........, ........) \)
3. (a) For the graph above, write the equation of the line, labelling clearly the key features

(b) On the above axes, draw the graph of the line that passes through (2,4) with a gradient of 2

4. (a) Write down the gradient and y-intercept of the line $4y = 2(8 - 4x)$

Gradient = ..................  \hspace{1cm} y\text{-intercept} = ..................

(b) A line has equation $y = -2x + \frac{3}{2}$. Give

(i) The equation of a line parallel to $L_1$

..................................................................................................................

(ii) The equation of a line perpendicular to $L_1$

..................................................................................................................
5.

(a) On the axes above, draw the graph of \( y = x^2 - 2x - 4 \)

(b) Use your graph to estimate

(i) The roots of \( x^2 - 2x - 4 \)

(ii) The minimum value of \( x^2 - 2x - 4 \)

(iii) The line of symmetry of \( x^2 - 2x - 4 \)
6.

On the axes above, are plotted three graphs:

(blue): \( y = -\frac{x}{6} \) and \( 2y - x = 2 \)
(red): \( y = x^2 - 4x + 2 \)

Use the above diagram to estimate the solutions to the pairs of simultaneous equations:

(a) \( y = -\frac{x}{6} \) and \( 2y - x = 2 \)

\[ \text{…………………………………………………………………………………………} \tag{2} \]

(b) \( y = \frac{x}{2} + 1 \) and \( y = x^2 - 4x + 2 \)

\[ \text{…………………………………………………………………………………………} \tag{2} \]
7.

(a) On the axes above, plot the graph of $y = \frac{1}{x} + 1$ (3)

(b) Using your graph, draw an appropriate line to estimate the solutions to the following equation

$$\frac{1}{x} - x + 1 = 0$$

............................................................................................................................................ (3)
8. For each of the following graphs, use one of the following descriptions and match so an equation

Linear
Quadratic
Cubic
Exponential
Reciprocal

\[ y = x^2 + 4x - 2 \]
\[ y = 4^x \]
\[ y = \frac{4}{x} \]
\[ y = x^3 + 4x^2 + x - 5 \]
\[ y = -x^3 \]
\[ y = x - 2 \]
\[ y = x^2 \]
\[ y = \frac{1}{2}x + 1 \]
Uni Pathways Year 2

Final Test

Name: ____________________
1.  

(a) N is the point (-3, -5). On the grid, mark the point N with a cross, and label it N.  
   \[ \text{1 mark for correctly identifying the point above} \]  

(b) Find the coordinates of the midpoint of the line NA  
   \[ (-\ldots, -\ldots) \]  

(c) Point M is lies along the line NB such that NM:MB is 1:2. Find the coordinates of M.  
   \[ (-\frac{5}{3}, -\ldots) \]  
   \[ \text{(allow decimals)} \]  
   \[ (1) \]
2.

(a) On the grid above, draw the graph of \( y = 1 - 2x \) from \( x = -3 \) to \( x = 3 \)

\[
\begin{array}{c|cccccccc}
 x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
 y & 7 & 5 & 3 & 1 & -1 & -3 & -5 \\
\end{array}
\]

1 mark: two mistakes in calculating values OR plotting points
2 marks: one mistake in calculating values OR plotting points
3 marks: line correctly plotted as above

(b) Using your graph, find the value of \( x \) when \( y = -1.5 \)

\[ x = 1.25 \]

(Allow values between 1.1 and 1.4) (1)

(c) \( f(x) = -2 + x \) is a function of \( x \). On the grid above, draw \( y = f(x) \)

\[
\begin{array}{c|cccccccc}
 x & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
 f(x) & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

1 mark: two mistakes in calculating values OR plotting points
2 marks: one mistake in calculating values OR plotting points
3 marks: line correctly plotted as above

(d) Using your graph, estimate the coordinates of the point of intersection between the lines drawn in parts (a) and (c)

\( (........, -1) \) (2)
3. (a) For the graph above, write the equation of the line, labelling clearly the key features
\[ y = -\frac{3}{2}x + 2 \] : 1 mark for the equation, 1 each for labelling the gradient and y-intercept

(b) On the above axes, draw the graph of the line that passes through (2,4) with a gradient of 2
1 mark for finding the equation \( y = -2x - 1 \)
1 mark for plotting the line above with 1 mistake in table of values/plotting
2 marks for no mistakes in plot

4. (a) Write down the gradient and y-intercept of the line \( 4y = 2(8 - 4x) \)

Gradient = \( \ldots \ldots \frac{2}{4} \) \quad y-intercept = \( \ldots \ldots \frac{4}{4} \) 
(1 mark each)

(b) A line has equation \( y = -2x + \frac{3}{2} \). Give
(i) The equation of a line parallel to \( L_1 \)
\[ y = -2x + [\text{any number}] \] 
(Allow decimals)

(ii) The equation of a line perpendicular to \( L_1 \)
\[ y = \frac{1}{2}x + [\text{any number}] \] 
(Allow decimals)
(a) On the axes above, draw the graph of $y = x^2 - 2x - 4$

(b) Use your graph to estimate

(i) The roots of $x^2 - 2x - 4$

-1.2 and -3.2 (1 mark each, allow ±0.1 on each) ..........................

(ii) The minimum value of $x^2 - 2x - 4$

-5 .............................................................................................................

(ii) The line of symmetry of $x^2 - 2x - 4$

$x = 1$ .............................................................................................................
On the axes above, are plotted three graphs:

(blue): \( y = -\frac{x}{6} \) and \( 2y - x = 2 \)
(red): \( y = x^2 - 4x + 2 \)

Use the above diagram to estimate the solutions to the pairs of simultaneous equations:

(a) \( y = -\frac{x}{6} \) and \( 2y - x = 2 \)

\((-1.5, 0.2)\)

\( \text{so } x = -1.5, y = 0.2 \) \(\text{ (1) mark for just (-1.5,0.2)}\) \(\text{ (allow ±0.1 on each)}\) \(\text{(2)}\)

(b) \( y = \frac{x}{2} + 1 \) and \( y = x^2 - 4x + 2 \)

\((0.25, 1.1)\)

\( \text{so } x = 0.25, y = 1.1 \) \(\text{ and }\)

\((4.25, 3.1)\)

\( \text{so } x = 4.25, y = 3.1 \) \(\text{ (2)}\)

(1) for each correct pair
Allow ±0.1 for each
7.

(a) On the axes above, plot the graph of \( y = \frac{1}{x} + 1 \) \hspace{1cm} (3)

<table>
<thead>
<tr>
<th>x</th>
<th>-1.6</th>
<th>-1.4</th>
<th>-1.2</th>
<th>-1</th>
<th>-0.8</th>
<th>-0.6</th>
<th>-0.4</th>
<th>-0.2</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
<td>0</td>
<td>-0.3</td>
<td>-0.7</td>
<td>-1.5</td>
<td>-4</td>
<td>-</td>
<td>6</td>
<td>3.5</td>
<td>2.7</td>
<td>2.3</td>
<td>2</td>
<td>1.8</td>
<td>1.7</td>
<td>1.6</td>
</tr>
</tbody>
</table>

1 mark for correct table of values, 2 marks for correct graph
(nb: mark for shape and points plotted: no straight-rulled lines)

(b) Using your graph, draw an appropriate line to estimate the solutions to the following equation

\[
\frac{1}{x} - x + 1 = 0
\]

\( x = -0.45, \quad \text{and} \quad x = 1.45 \) \hspace{1cm} (3)

1 mark each, allow \( \pm 0.05 \)
8. For each of the following graphs, use one of the following descriptions and match so an equation

- Linear
  - \( y = x^2 + 4x - 2 \)
- Quadratic
  - \( y = 4^x \)
- Cubic
  - \( y = \frac{4}{x} \)
- Exponential
  - \( y = x^3 + 4x^2 + x - 5 \)
- Reciprocal
  - \( y = -x^3 \)
  - \( y = x - 2 \)
  - \( y = x^2 \)
  - \( y = \frac{1}{2}x + 1 \)

(i) \( y = 4^x \)  \( \text{Exponential} \)  \( \text{(1)} \)
(ii) \( y = x - 2 \)  \( \text{linear} \)  \( \text{(1)} \)
(iii) \( y = \frac{4}{x} \)  \( \text{Reciprocal} \)  \( \text{(1)} \)
(iv) \( y = x^2 + 4x - 2 \)  \( \text{quadratic} \)  \( \text{(1)} \)
(v) \( y = x^3 + 4x^2 + x - 5 \)  \( \text{cubic} \)  \( \text{(1)} \)