Circle Theorems

Pupil Name

Handbook Designed by Dr. Niccolo' Salvatori, Ph. D.

Timetable

Timetable – Tutorials

Tutorial	Date	Time	Location
Baseline	**09/01/2019**	**4 pm**	**F.2.7**
1	**16/01/2019**	**4 pm**	**F.2.7**
2	**23/01/2019**	**4 pm**	**F.2.7**
3	**30/01/2019**	**4 pm**	**F.2.7**
4	**06/02/2019**	**4 pm**	**F.2.7**
Final Assignment	**13/01/2019**	**4 pm**	**F.2.7**
Feedback	**27/02/2019**	**4 pm**	**F.2.7**

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- Assessment & Extension C P12
- Activity D P14
- Assessment & Extension D P14

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Course Rationale

The main objective of this course is to acquire a deep understanding of the circle theorems, especially in order to be able to connect them all and solve advanced problems. To fulfil this goal this workbook is composed by 4 tutorials aiming to cover all the 7 Circle Theorems. This work has been created to guide you through this content by increasing the complexity of the tasks involved.

In order to access all the desired skills, these tutorials have been developed using the principle of "deliberate practice", an approach to learning that aims at practising specific skills at a time by tackling problems with increasing difficulty but similar structure. In this way, the individual variations of each problem should disclose to you, the learner, what are the key points of the exercise (and in our case, what are the key points of the theorems).

Proofs are the spine of Mathematics and all the branches that hold our understanding of the structure of the Universe are rooted into mathematical proofs. This is why all Circle Theorems proofs are shown in this workbook. It has been an ancient pursuit the one to represent our world in terms of geometrical problems with exact solutions. Since before the Greeks humans have reasoned and taught through geometrical proofs. Thanks to the contributions of geniuses such Thales or Pythagoras, who were following such logical path, we can now appreciate how the many properties of shapes fit together and explain the our reality.

We hope that throughout this course you will discover the pleasure of solving geometrical problems and learn the importance of sound thinking, as exemplified by the Circle Theorems and their contribution to our learning.

Glossary of Keywords

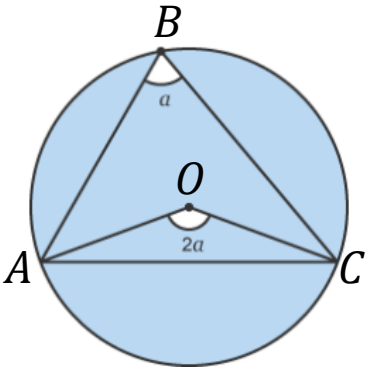
Word	Definition	In a sentence
Bisect	To divide into two equal sections; to cut in half.	
Centre	The point that is equally distant from every point on the circumference of a circle (or sphere)	
Chord	A straight line segment joining two points on a circle	
Circumference	Name of the perimeter of a complete circle, that is, the distance all around it	
Congruent	Two shapes are "congruent" when they are exactly the same shape and exactly the same size	
Hypotenuse	The longest side of a right-angled triangle, which is opposite the right angle	
Isosceles	Triangle with two equal sides (equal length). Angles opposite the equal sides are equal.	
Pythagoras's theorem	Pythagoras's theorem applies to right-angled triangles. The area of the square drawn on the hypotenuse is equal to the sum of the squares drawn on the other two sides.	
Quadrilateral	A quadrilateral is a shape with four straight sides and four angles.	
Radius	The distance from the centre of a circle to its circumference. The plural of radius is radii	
Subtended	An angle created by an arc or chord at a given point	
Tangent	A straight line that just touches a point on a curve. A tangent to a circle is perpendicular to the radius which meets the tangent	
Theorem	A mathematical statement that can be logically proved to be true.	

Tutorial 1 – Angle at the centre and circumference; Angels in the same segment

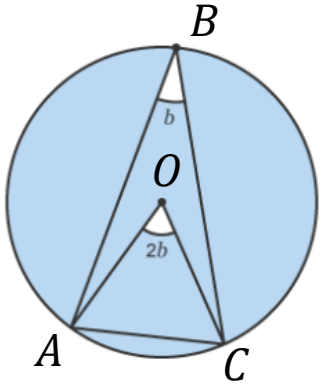
Objective A. Angles at the centre and circumference
Keywords: theorem, centre, circumference, subtended….

Objective A: understanding and applying the relationship between the angle subtended by an arc at the centre and the angle subtended by an arc at the circumference

Theorem A: The angle subtended by an arc at the centre is twice the angle subtended by the same arc at the circumference



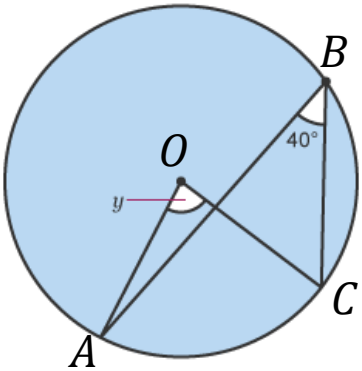
$$\widehat{AOC} = 2\widehat{ABC}$$



Simply put, the angle at the centre is double the angle at the circumference.

Example A

Worked example 1: Calculate the missing angle y.

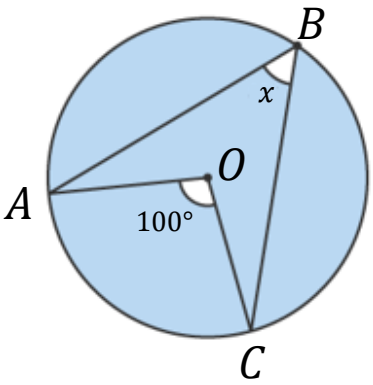


$\widehat{AOC} = y$
$\widehat{ABC} = 40°$

Theorem A states that $\widehat{AOC} = 2\widehat{ABC}$, therefore

$$x = 2 \times 40° = 80°$$

Worked example 2: Calculate the missing angle x.



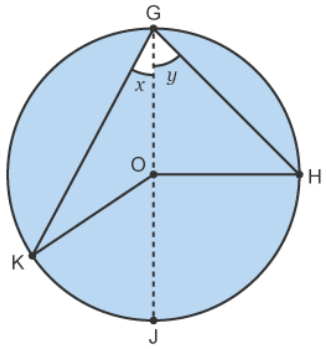
$\widehat{ABC} = y$
$\widehat{AOC} = 100°$

Theorem A states that $\widehat{AOC} = 2\widehat{ABC}$, therefore

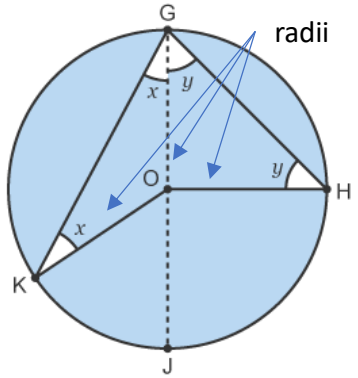
$$x = 100° \div 2 = 50°$$

Proof of Theorem A:

Let $\widehat{KOH}$ be the angle at the centre subtended by the arc $\overset{\frown}{KH}$ and $\widehat{KGH}$ be the corresponding angle at the circumference.
If we draw the diameter $\overline{GJ}$, then we can consider two isosceles triangles, KGO and GHO, with equal sides OK, OG and OH, which are all radii. Set $\widehat{KGO} = x$ and $\widehat{HGO} = y$. Therefore, $\widehat{KGH} = x + y$.

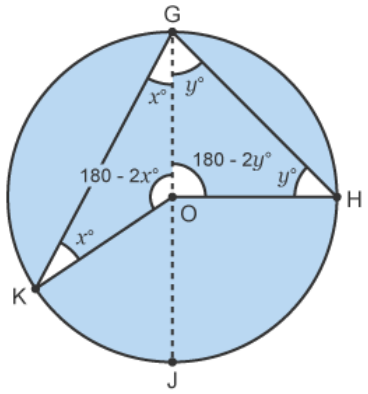


Then, since the triangles KGO and GHO are isosceles, we can conclude that $\widehat{GKO} = x$ and $\widehat{GHO} = y$.

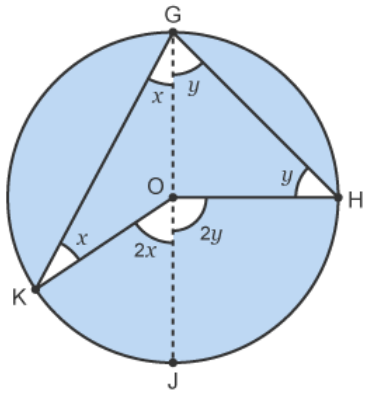


Remember: in an isosceles triangle, the angles opposite to the equal sides are equal.

Now me must remember that the angles of a triangle always add up to $180°$. With this fact in mind, we can conclude that $\widehat{KOG} = 180° - 2x$ and $\widehat{GOH} = 180° - 2y$.



Finally, $\widehat{KOJ} = 2x$ and $\widehat{HOJ} = 2x$ because angles on a straight line always add up to $180°$. As the figure on the side shows, we have reached the conclusion that the angle at the centre, $\widehat{KOH}$, is equal to $2x + 2y = 2(x + y) = 2\widehat{KGH}$, which means that the angle at the centre is twice the angle at the circumference: $\widehat{KOH} = 2\widehat{KGH}$.



Practice A

1) In the diagram, O is the centre of the circle. A, B and C are points on the circle.

Find the size of angle CBA.

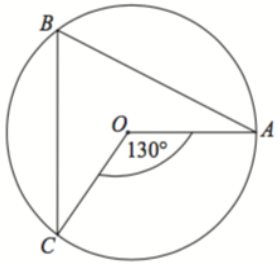


Diagram **NOT** accurately drawn

2) A , B and C are points on the circumference of a circle, centre O . AC is a diameter of the circle.

Give a reason why angle $ABC = 90^{\circ}$.

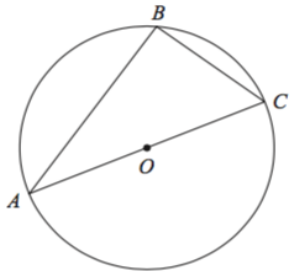


Diagram **NOT** accurately drawn

3) L , M and P are points on a circle, centre O.

Angle $LMP = 48^{\circ}$.

Write down the size of angle LOP

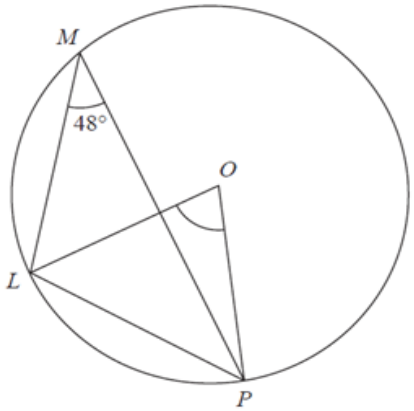


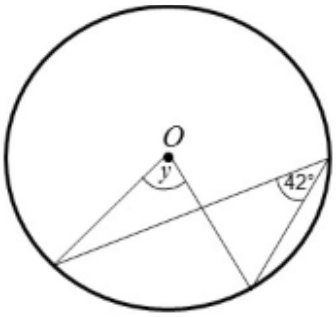
Diagram **NOT** accurately drawn

Assessment A

The diagram shows a circle, centre O.

Work out angle y.

Not drawn accurately



A	B	C	D
42°	84°	138°	21°

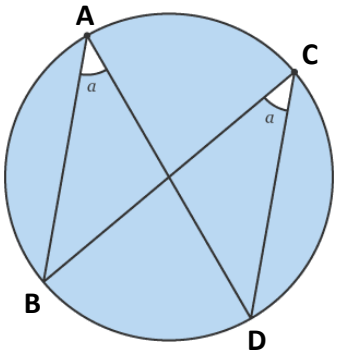
Objective B. Angles in the same segment

Keywords: circumference, subtended

Objective B: understanding and applying the relationship between the angles at the circumference subtended by the same arc

Theorem: the angles at the circumference subtended by the same arc are equal, or, angles in the same segment are equal

$$\widehat{BAD} = \widehat{BCD}$$



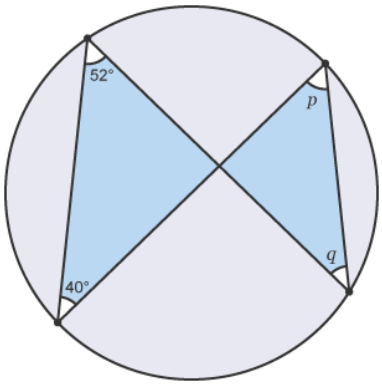
Example B

Calculate angles p and q

Angles in the same segment are equal

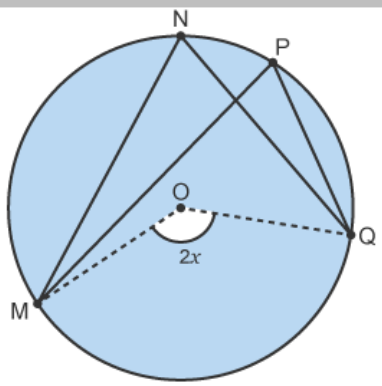
$p = 52°$

$q = 40°$



Proof of Theorem B

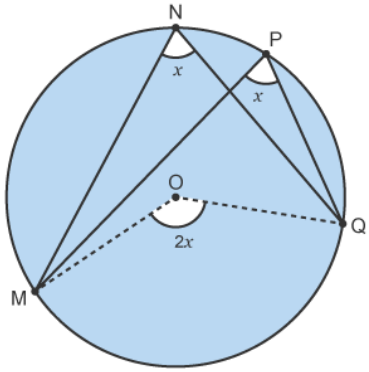
Let the obtuse angle $\widehat{MOQ} = 2x$



Using the circle theorem (Theorem A), the angle is twice the angle at the circumference.

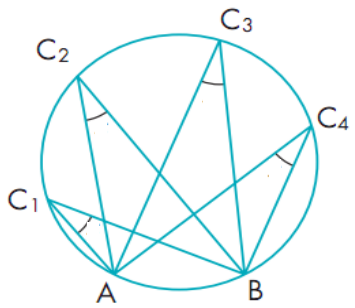
Angle $\widehat{MNQ} = x$ and angle $\widehat{MPQ} = x$

Therefore $\widehat{MNQ} = \widehat{MPQ}$

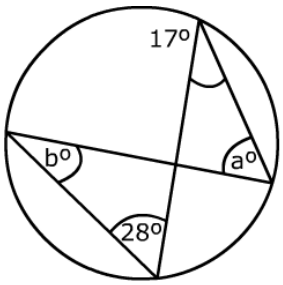


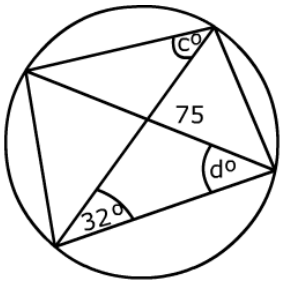
Practice B

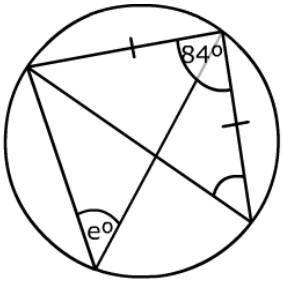
1) What can you tell me about the angles in this circle?
Knowing that the angles add up to 128, work out each angle degree.



2) Which angle and triangle facts are needed in the following?
Work out the unknown angles : a, b, c, d and e.



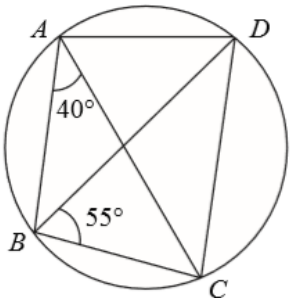




Assessment B

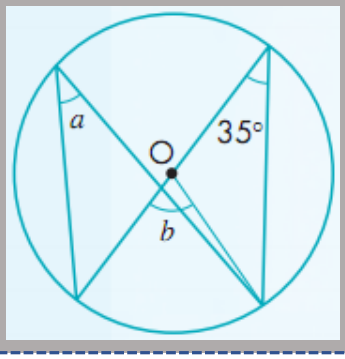
A, B, C and D are points on a circle. Angle $\widehat{BAC} = 40°$ and $\widehat{DBC} = 55°$

Find the size of angle $\widehat{DAC}$



A	40°	B	80°	C	55°	D	110°

Extension: Identify and justify which circle theorems you would need to find a and b



Tutorial 2 – Cyclic quadrilaterals; Angles in a semicircle

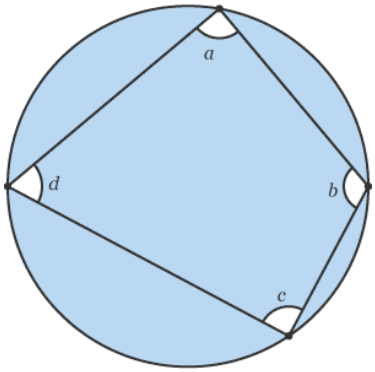
Objective C. Cyclic quadrilaterals
Keywords: quadrilateral

Objective C: understanding and applying the relationship between the angles of a cyclic quadrilateral

Theorem: The opposite angles in a cyclic quadrilateral add up to 180°

$a + c = 180°$

$b + d = 180°$



Study tip
A cyclic quadrilateral is a quadrilateral drawn inside a circle, which every vertex must touch the circumference of the circle

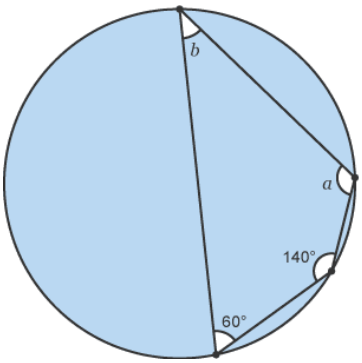
Example C

Calculate the angles *a* and *b*.

The opposite angles in a cyclic quadrilateral add up to 180°

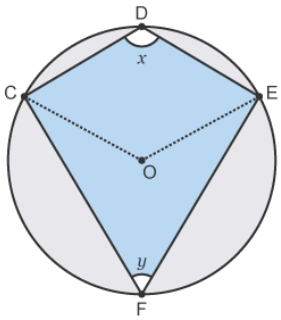
$a + 60° = 180°$ $\quad a = 180° - 60° = 120°$

$140° + b = 180°$ $\quad b = 180° - 140° = 40°$



Proof of Theorem C

Let angle $\widehat{CDE} = x$ and angle $\widehat{EFC} = y$



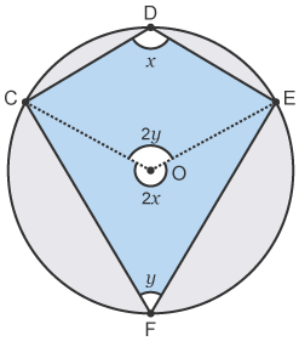
The angle at the centre is twice the angle at the circumference (Theorem A)

The reflex angle $\widehat{COE} = 2x$ and the angle $\widehat{COE} = 2y$

Angles around a point add up to 360°

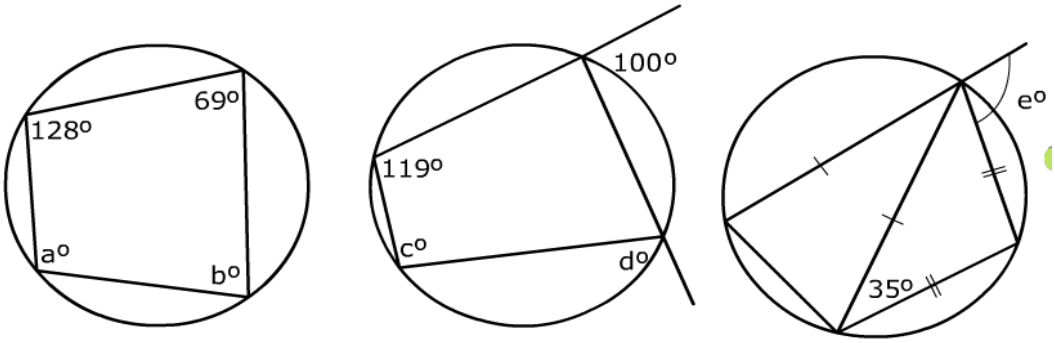
$2x + 2y = 360°$

So $x + y = 180°$



Practice C

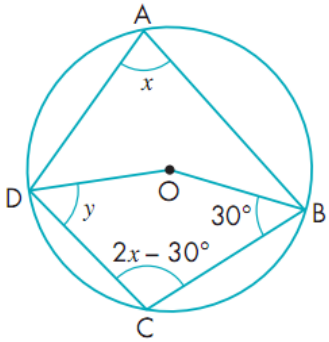
1) Work out the missing angles



2) In the diagram, O is the centre of the circle.

a) Explain why $3x - 30° = 180°$

b) Work out the size of $\overset{\frown}{CDO}$, give reasons in your working

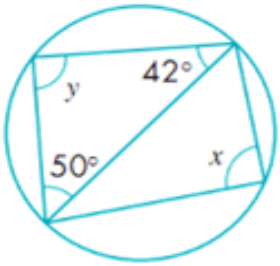


Study hint
Revise previous circle theorems

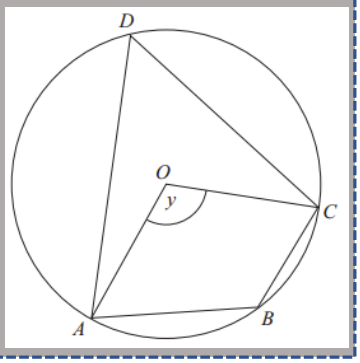
Assessment C

Find the angles x and y

A	x=88° y=88°	B	x=92° y=88°	C	x=88° y=92°	D	x=90° y=90°



Extension: Identify and show the working out to find the angle $\overset{\frown}{ABC}$ in terms of y.



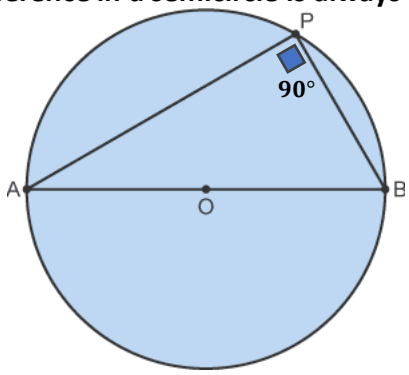
Objective D. Angles in a semicircle

Keywords: theorem, centre….

Objectives D: understanding and applying the fact that the angle at the circumference in a semicircle is always a right angle, i.e. $90°$.

Theorem D: The angle at the circumference in a semicircle is always a right angle, i.e. $90°$.

$\widehat{APB} = 90°$.



Example D

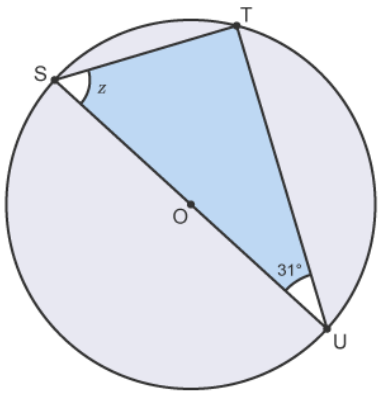
Worked example 1: calculate the angle z.

By the theorem, the angle at the circumference in a Semi circle is $90°$, i.e. $\widehat{STU} = 90°$.

Since the angles in a triangle add up to $180°$, we have

$z + 31° + 90° = 180°$

Therefore, $z = 180° - 31° - 90° = 59°$.

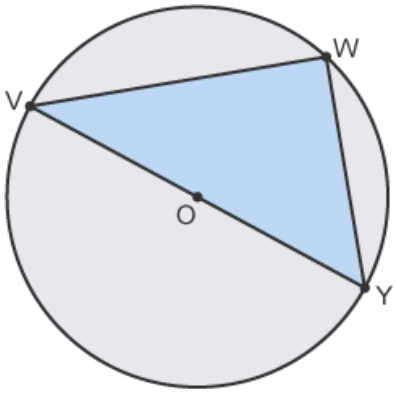


Proof of Theorem D

Since $\overline{VU}$ is the diameter of the circle, the angle $\widehat{VOU}$ is the angle on a straight line, which implies that $\widehat{VOU} = 180°$.

Now, $\widehat{VWU}$ is an angle at the centre that is subtended by the same arc that subtends the angle at the centre $\widehat{VOU}$. Therefore, $\widehat{VOU} = 2\widehat{VWU}$ by Theorem A.

This implies $\widehat{VWU} = \frac{1}{2}\widehat{VWU} = \frac{1}{2} \times 180° = 90°$.



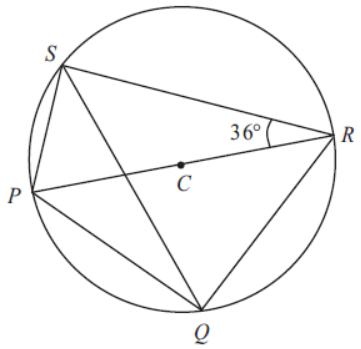
Attention!
Do not forget the other theorems that you have seen so far can also be used!

Practice D

Exercise 1:

P, Q, R and S are points on a circle, centre C.
PCR is a straight line.
Angle $PRS = 36°$.

Calculate the size of angle RQS.
Give a reason for each step in your working.

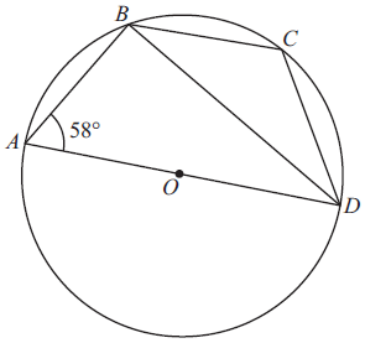


Exercise 2:

A, B, C and D are four points on a circle, centre O.
AD is a diameter of the circle.
Angle $BAD = 58°$

(a) Calculate the size of angle ADB.
(b) (i) Calculate the size of angle BCD.
(ii) Give a reason for your answer.

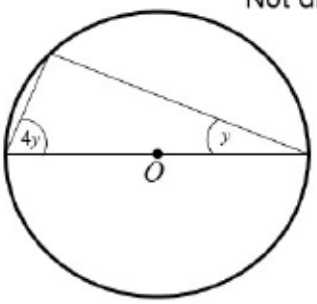
Study tip
Do not forget to stay on the look for the other theorems as well!



Assessment D

The diagram shows a circle, centre O.

Not drawn accurately

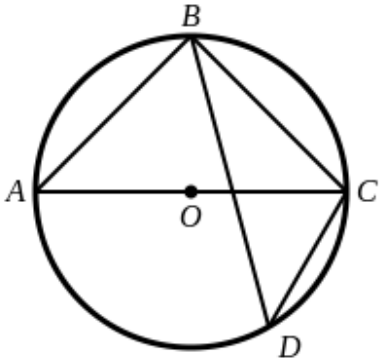


What is the value of y?

A	B	C	D
36°	18°	72°	4.7°

Extension:

In the diagram below, angle ACB = 40 °. What is the value of BDC?



A	B	C	D
30°	40°	50°	45°

Tutorial 3 – Tangents and circles; Chords

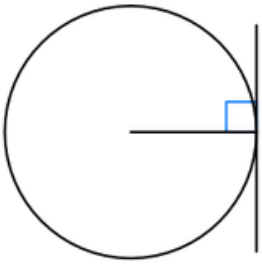
Objective E. Tangents and circles

Keywords: tangent, radius, isosceles, quadrilateral, congruent

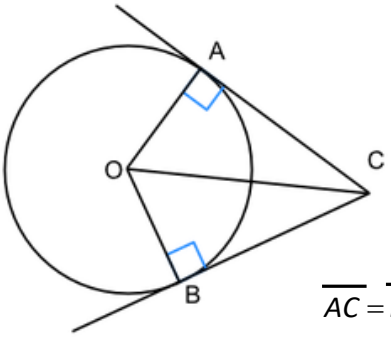
Objective E: understanding and applying the relationship between a tangent and a radius and the relationship between tangents at the same point

Theorem: a) The angle between a tangent and a radius is 90°; b) tangents which meet at the same point are equal in length.

a)



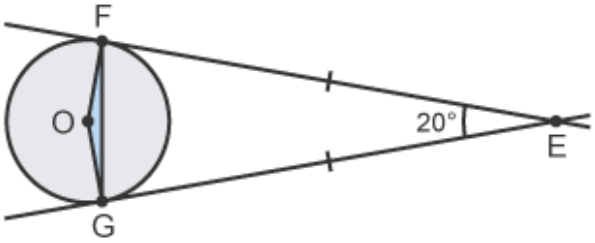
b)



$\overline{AC} = \overline{BC}$

Example E

Calculate the angles $\widehat{EFG}$ and $\widehat{FOG}$



Triangle $\widehat{GEF}$ is an isosceles triangle.

$$\widehat{FGE} = \widehat{EFG} = \frac{180° - 20°}{2} = 80°$$

The angle between the tangent and the radius is 90°

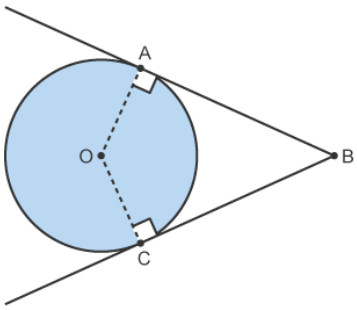
$$\widehat{EFO} = \widehat{EGO} = 90°$$

FOGE is a quadrilateral. The angles in a quadrilateral add up to 360°

$$\widehat{FOG} = 360° - 90° - 90° - 20° = 160°$$

Proof of Theorem E

The angle between the tangent and the radius is 90°; $\widehat{BCO} = \widehat{BAO}$



$\overline{AO}$ and $\overline{OC}$ are both radii of the circle, they have the same length

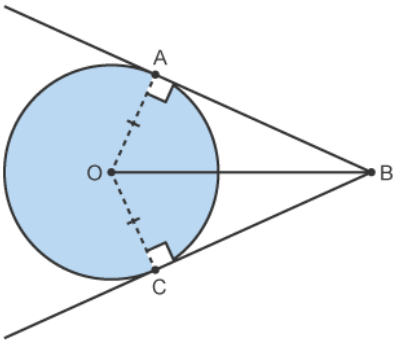
Study hint
Revise congruent definition or the RHS rule

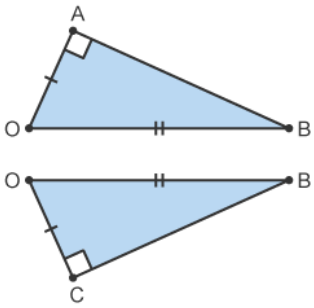
Drawn the line OB. It creates two congruent triangles: OCB and OAB.

$\overline{OB}=\overline{OB}$, $\overline{OC}=\overline{OA}$ and $\widehat{OCB}=\widehat{OCB}$

Congruent triangles are identical.

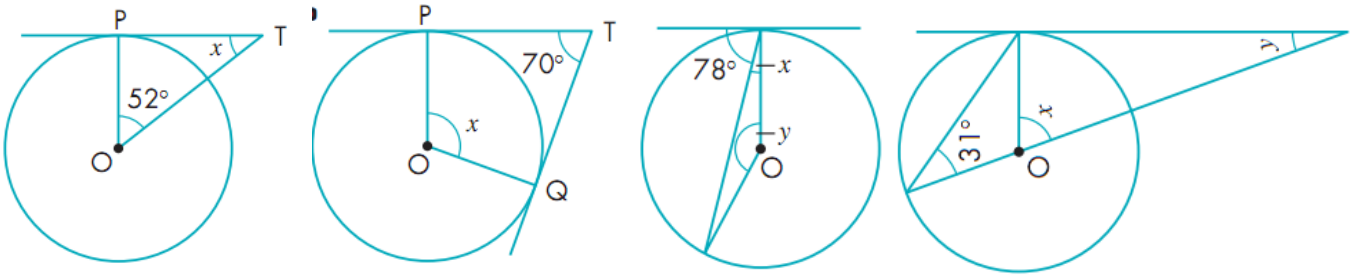
So $\overline{CB}=\overline{AB}$



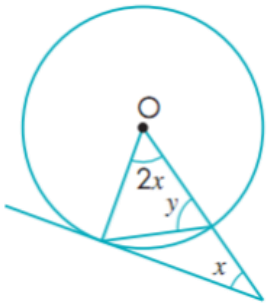


Practice E

1) Work out the unknown variables.



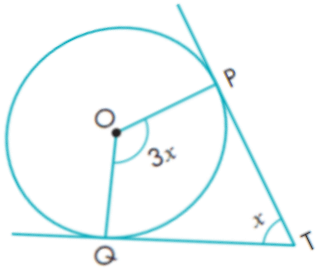
2) Calculate *x* and *y*. Show your working out.



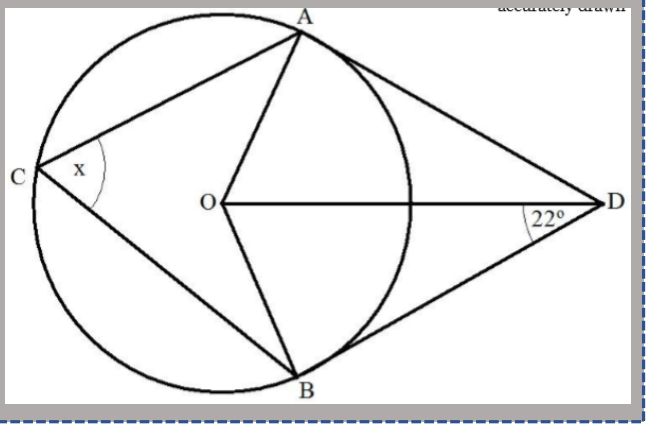
Assessment E

Work out the value of *x*

A	45°	B	22.5°	C	135°	D	90°



Extension: List the circle theorems you need to calculate x. Show the process to work out x.

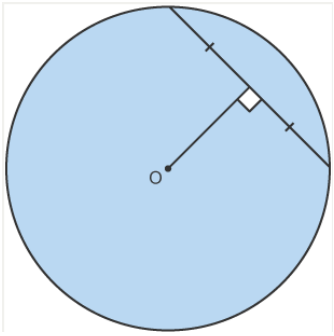


Objective F. Chords

Keywords: bisect, chord, hypotenuses, midpoint, perpendicular, Pythagoras' theorem , radius

Objective F: understanding and applying the relationship between a chord and the perpendicular from the centre of a circle

Theorem: the perpendicular from the centre of a circle to a chord bisects the chord



Example F

A circle has a radius of 5 cm. The chord $\overline{EF}$ is 7 cm. How far is the midpoint of the chord from the centre of the circle?

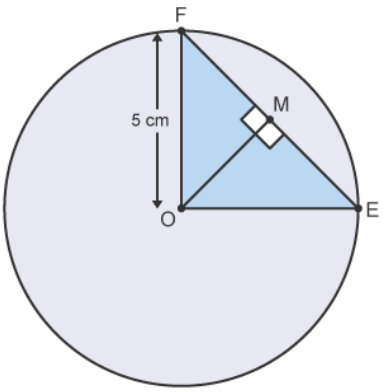
The two radii $\overline{OE}$ and $\overline{OF}$ are the hypotenuses of 2 right-angled triangles.

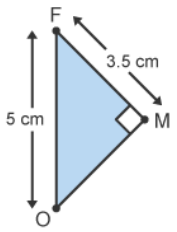
The distance $\overline{FM}$ is half the length of the chord. (The perpendicular from the centre of a circle to a chord bisects the chord.)

$\overline{FM}$ = 3.5 cm

Use Pythagoras's theorem to calculate the length $\overline{OM}$.

$$\overline{OF}^2 = \overline{FM}^2 + \overline{OM}^2$$
$$5^2 = 3.5^2 + OM^2$$
$$\overline{OM}^2 = 12.75$$
$$\overline{OM} = 3.6cm$$





Proof of Theorem F

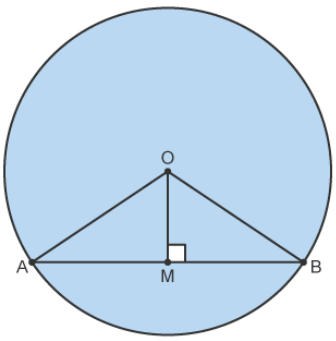
In the diagram below, $\overline{AB}$ is the chord of a circle with centre O. $\overline{OM}$ is the perpendicular from the centre to the chord.

Look at triangles OAM and OBM. The hypotenuses ($\overline{OA}$ and $\overline{OB}$) are same, As they are both the radius of the circle.

$\overline{OM}$ is common to both triangles. OMA and OMB are both right angles.

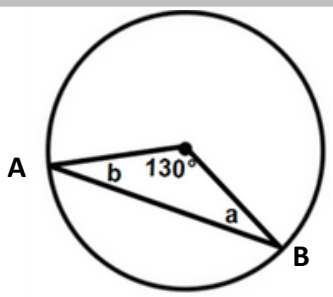
Triangles OAM and OBM are congruent (RHS), so it follows that $\overline{AM}$ = $\overline{MB}$.

Therefore, M is the midpoint of $\overline{AB}$, and the chord has been bisected.

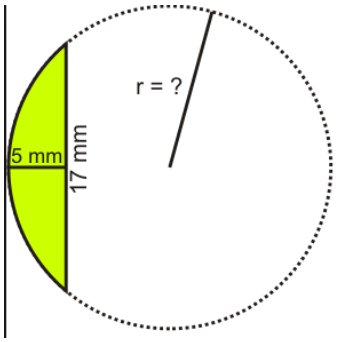


Practice F

1) Work out the length of the $\overline{AB}$ chord

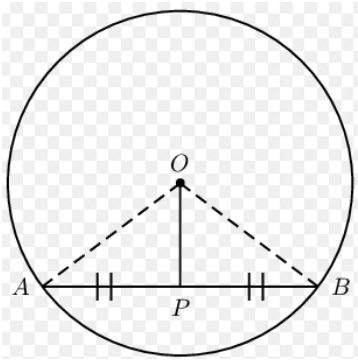


2) Calculate the radii of the following circle.



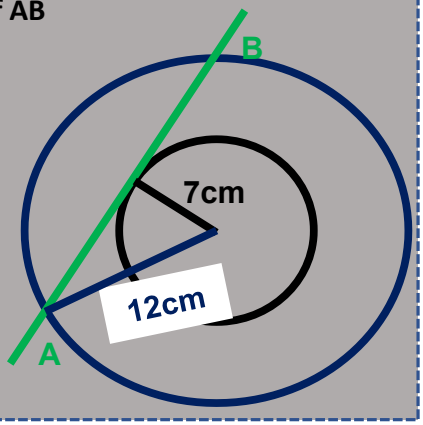
Assessment F

Which theorem would you use to obtain the length of $\overline{OP}$?



A	Tangent	B	Chords	C	Pythagoras'	D	Angles at the centre and circumference

Extension: Two circles with the same centre have a radii of 7 cm and 12 cm respectively. A tangent to the inner circle cuts the outer circle at A and B. Find the length of AB

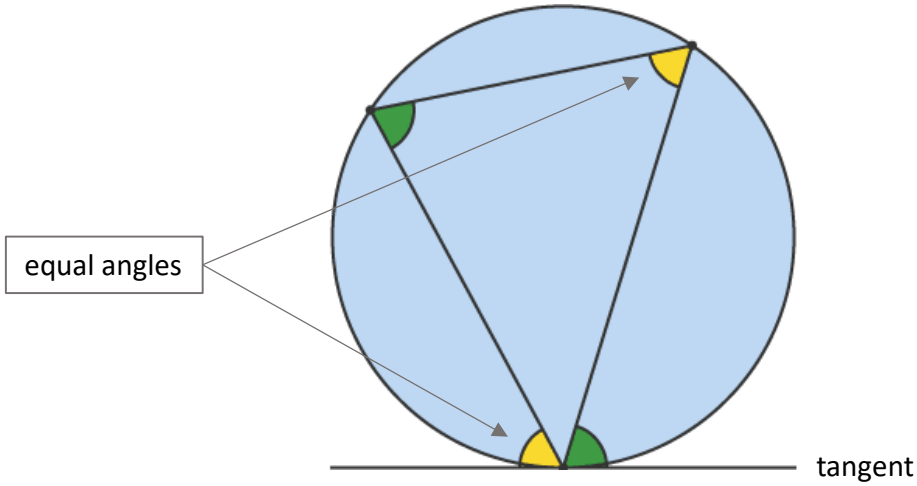


Tutorial 4 – Alternate Segments; Revision

Objective G. Alternate segments
Keywords: theorem, centre….

Objectives G: understanding and applying the relationship between the angles obtained from a tangent and a chord

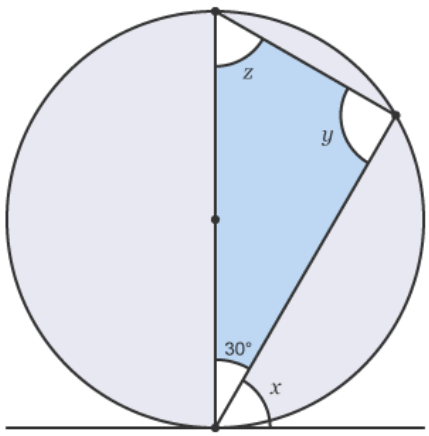
Theorem G: The angle between a tangent and a chord is equal to the angle in the alternate segment.



Example G

Worked example:

Calculate the missing angles x, y and z.



y is the angle is a semicircle, so $y = 90°$ by Theorem D.

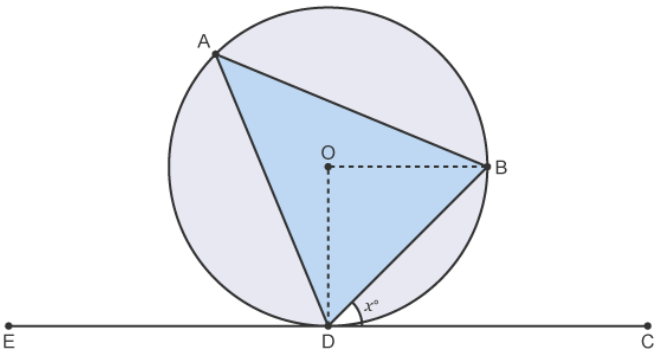
Study tip
Revise the past theorems, especially Theorem D!

Also, since the angles in a triangle add up to $180°$, we have $z = 180° - 30° - 90° = 60°$.

Finally, $x = z$ because of Theorem G, as x is the angle between the tangent and the chord and z is the angle in the alternate segment. Thus, $x = 60°$.

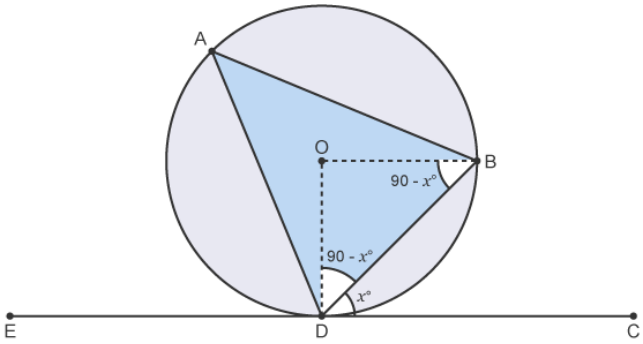
Proof of Theorem G

Let us consider the angle $\widehat{BDC}$, which is defined by the tangent $\overline{EC}$ and the chord $\overline{DB}$ and set $\widehat{BDC} = x$. Let also draw the radii $\overline{OD}$ and $\overline{OB}$, which imply that the triangle OBD is isosceles.



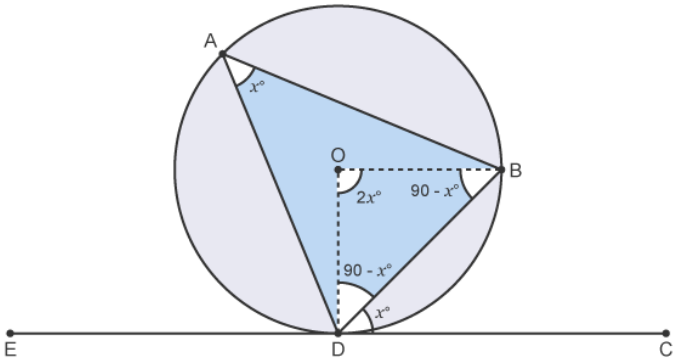
By Theorem E we know that the radius $\overline{OD}$ and the tangent $\overline{EC}$ are perpendicular because they meet at the point D. Thus, the angle $\widehat{ODB}$ must be equal to $90° - x$.

Since the triangle OBD is isosceles, we must also have $\widehat{OBD} = 90° - x$.



Again, the angles in a triangle add up to $180°$, so $\widehat{DOB} = 180° - (90° - x) - (90° - x) = 2x$.

Finally, $\widehat{DAB}$ is an angle at the circumference subtended by the arc that subtends the angle at the centre $\widehat{DOB} = 2x$, which Theorem A tells us to be twice $\widehat{DAB}$. Thus, in conclusion $\widehat{DAB} = \frac{1}{2}\widehat{DOB} = \frac{1}{2} \times 2x = x$, as requested.



Practice G

Exercise 1:

A, B, C and D are points on a circle.
PA is the tangent to the circle at A.
Angle $PAD = 54°$, angle $ACB = 36°$ and angle $ADC = 78°$.

(a) (i) Find the size of angle ACD.

(ii) Give a reason for your answer. **(2)**

(b) Explain why BD is a diameter of the circle. **(2)**

(c) (i) Work out the size of angle ABC.

(ii) Give a reason for your answer. **(2)**

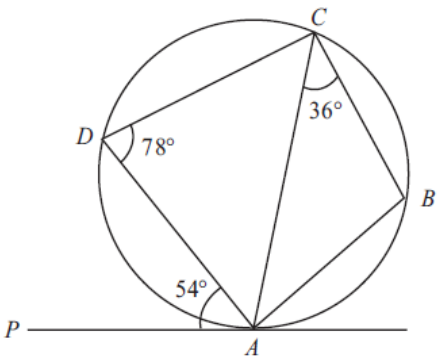
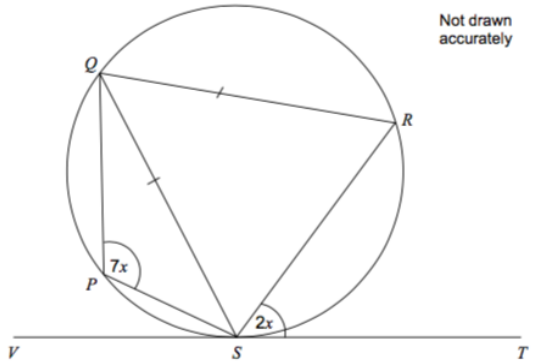


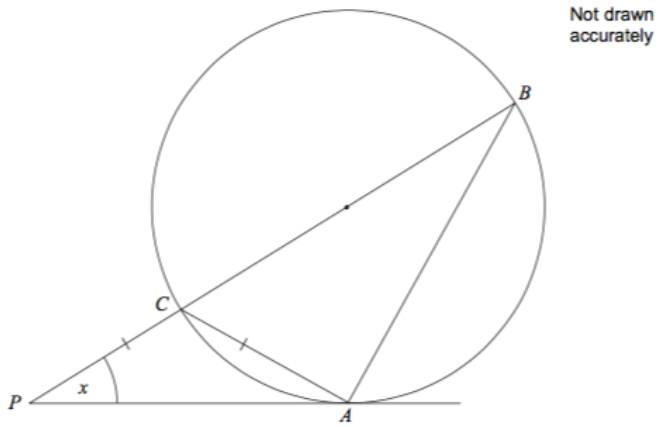
Diagram **NOT** accurately drawn

Exercise 2:

$PQRS$ is a cyclic quadrilateral.
$QS = QR$
VST is a tangent to the circle.
Work out the value of x .

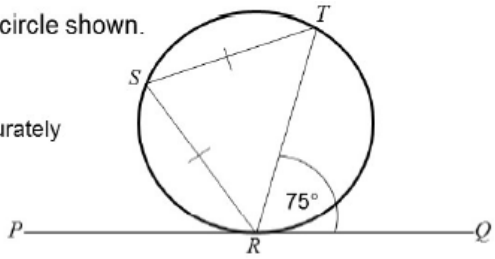


Exercise 3:
A, B and C are points on the circumference of a circle.
BC is a diameter,
BCP is a straight line,
AP is a tangent to the circle.
$PC = CA$
Work out the value of angle CPA,
marked x on the diagram.



Assessment G

PQ is a tangent to the circle shown.



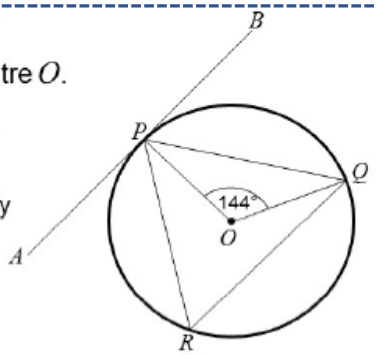
Calculate angle RTS.

A	B	C	D
75°	30°	52.5°	105°

Study tip
Remember to provide a motivation for your answer

Extension:

The diagram shows a circle, centre O.
AB is a tangent to the circle at P.



Work out angle BPQ.

A	B	C	D
144°	72°	45°	18°

Revision Techniques & Final Assignment

Feedback tutorial

What is the Purpose of this Tutorial?

- To receive feedback on final assignments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.

To reflect on the programme including what was enjoyed and what was challenging.

Final assignment feedback

What I did well...	What I could have improved on...

My target for future work is...

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?

What did you find challenging about the programme?	How did you overcome these challenges?

Notes

Notes

Notes

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