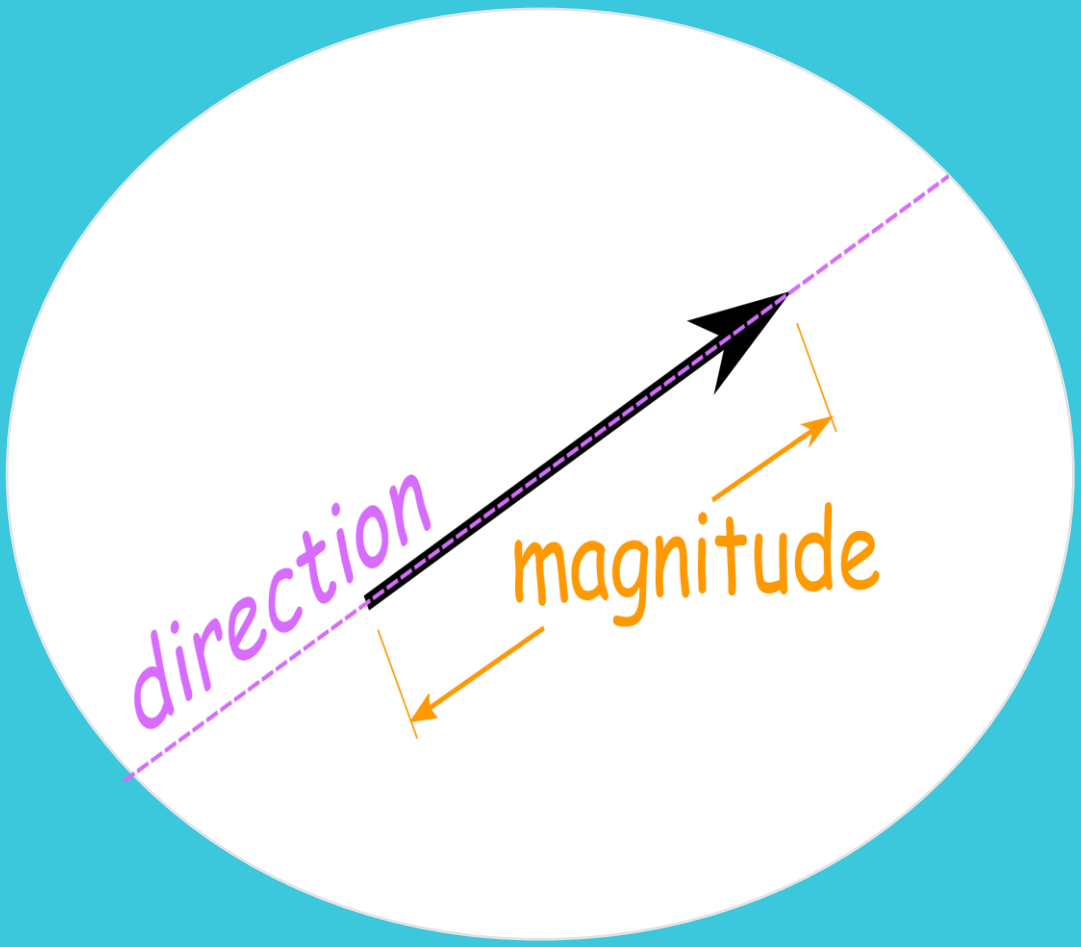


Combining 2D vectors

Key Stage 4 Revision Programme

Pupil Name	
Handbook Designed by	Sam Kelly

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1 (Baseline)			
2			
3			
4			
5			
6 (Assessment)			
7 (Feedback)			

Contents

Course Rationale	P3
Tutorial 1: Vectors – Direction and Magnitude	P4
Tutorial 2: Vector Notation	P11
Tutorial 3: Vectors in Geometry	P18
Tutorial 4: Synoptic questions	P24
Feedback and Reflection	P28

Course Rationale

Vectors are a mathematical concept that feature heavily in mathematics and applied mathematics courses, such as physics and engineering. Understand vectors and their uses will make many other topics easier to understand, as well as ensure better grades in a GCSE exam!

Vectors tend to feature more heavily in the new mathematics GCSE syllabus, and are a topic that students routinely find difficult to grasp. This booklet aims to make vectors more accessible starting from their basic uses in translation and enlargement of shapes, through more difficult geometrical problems and finally in the more challenging synoptic style questions that students aiming for grade 7+ would be expected to answer.

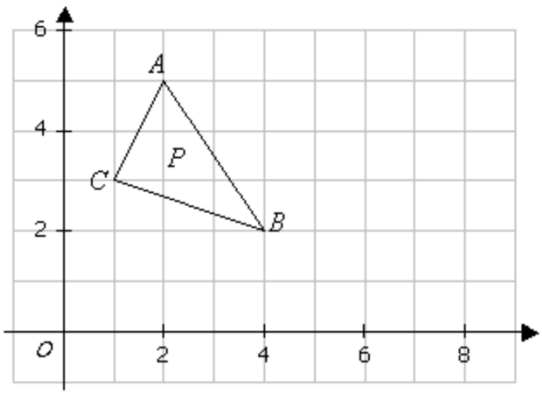
Tutorial 1: Vectors – Direction and magnitude.

Objectives
1) What is a vector?
2) Drawing and interpreting vectors (translation and enlargement)
3) Scalar multipliers and addition of vectors

Part 1 - What is a vector?

Starter

The triangle P is mapped onto the triangle Q by the translation $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.



a) Find the coordinates of triangle Q.

b) On the diagram, draw and label triangle Q.

In this question you are expected to recognise and use a column vector, $\binom{4}{-1}$. This notation is a concise way of telling an individual how far to move and in which direction, what does it mean? (Take care with the sign!)

You are all familiar with scalar quantities as something to describe an amount. Money, time and mass are all scalar quantities. You can tell someone how much money you have e.g. £20, but to give money a direction doesn't really make sense. You can't tell someone you have £20 due north for example!

Vectors are different from scalars in that they tell someone not only an amount, but also a direction. Let's look at perhaps the most obvious example of this, displacement. If you are to walk between 2 points, say A and B, it's important to specify how far you need to go, but also in which direction, otherwise you could easily end up at some other point. In addition, the journey between B and A is NOT the same as the journey between A and B. The distance between the 2 points is obviously the same, however the direction has changed.

A vector is also allowed to move around in space. A vector on its own does not have a starting point, it ONLY tells an individual how far to move in a particular direction. This is best illustrated with an example…

Towns P and Q are 5 km apart, as are towns X and Y, Q lies due east of P and Y lies due east of X. The vector describing the journey from P to Q is the same as the one describing the journey form X to Y, they are the same distance apart and in the same direction, BUT the start (and therefore end) point is different!

Practice questions

1. Which of these are vector quantities and which are scalar?
   a) Velocity
   b) Speed
   c) Area
   d) Volume
   e) Acceleration
   f) Force

2. A man walks 3 miles from a town called Acton to another town called Burton. After 2 hours he walks directly back to Acton. Is the vector that maps each journey the same? (Explain your answer)

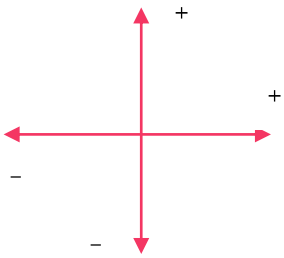
3. ABCD is a rectangle

   a. Is the vector that maps the journey from A to B the same as the one that maps D to C?

   b. Is the vector that maps A to D the same as the one that describes the journey from C to B?

Part 2 - Drawing and interpreting vectors

Vectors are often easiest to recognise when they are drawn on a grid, with little effort one can see how far we need to move and in which direction. If you remember, the top number in a column vector instructs how far to move horizontally, and the bottom number instructs how far to move vertically. A positive number means move right or up, a negative number means move left or down, see the figure below for a reminder.



If you remember the column vector from the starter question, we had to...

Move 4 spaces to the right → $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ ← Move 1 space down

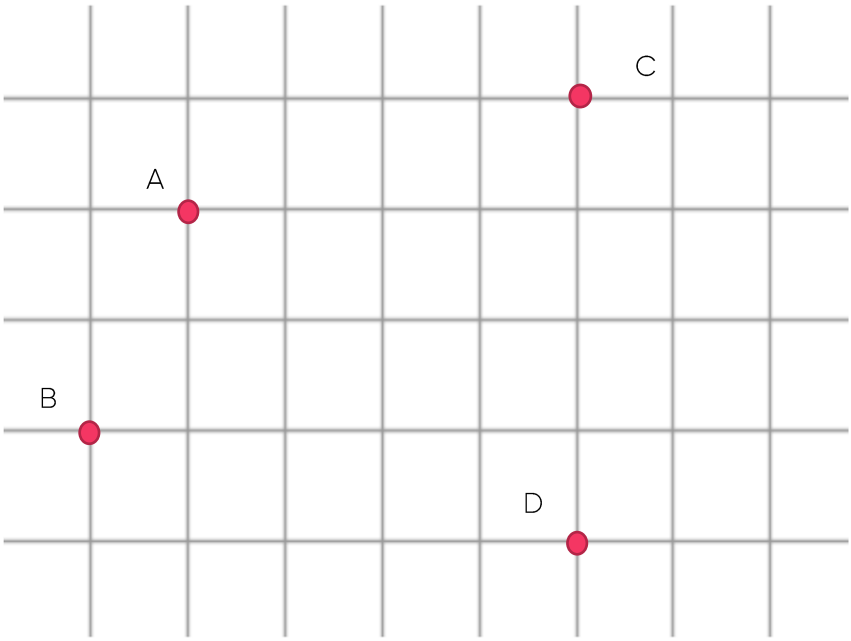
On the grid below, write the column vectors that describe the following journeys.

i. A to B

ii. C to D

iii. D to A

iv. C to B



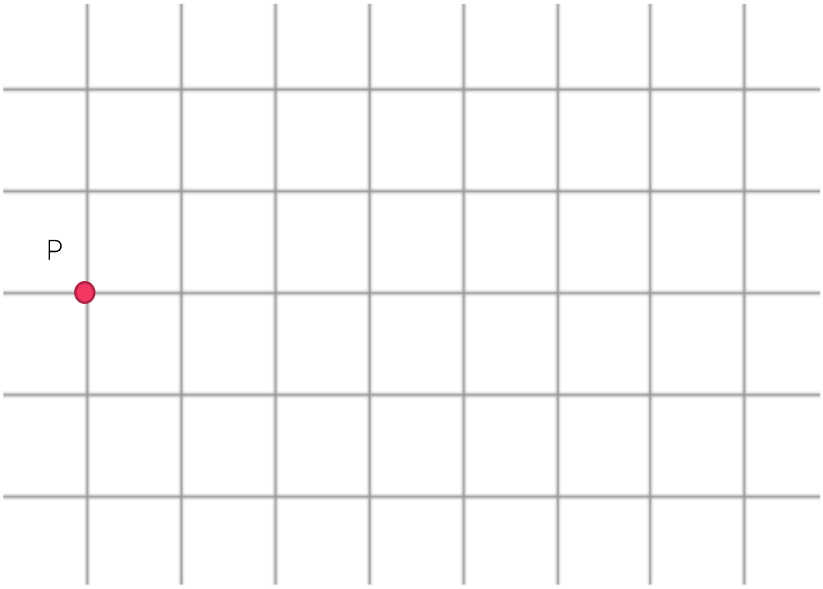
On the grid below, starting from the point labelled and continuing from the end of the one previously drawn, draw the following vectors.

i. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ii. $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

iii. $\begin{pmatrix} -6 \\ -2 \end{pmatrix}$

iv. $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

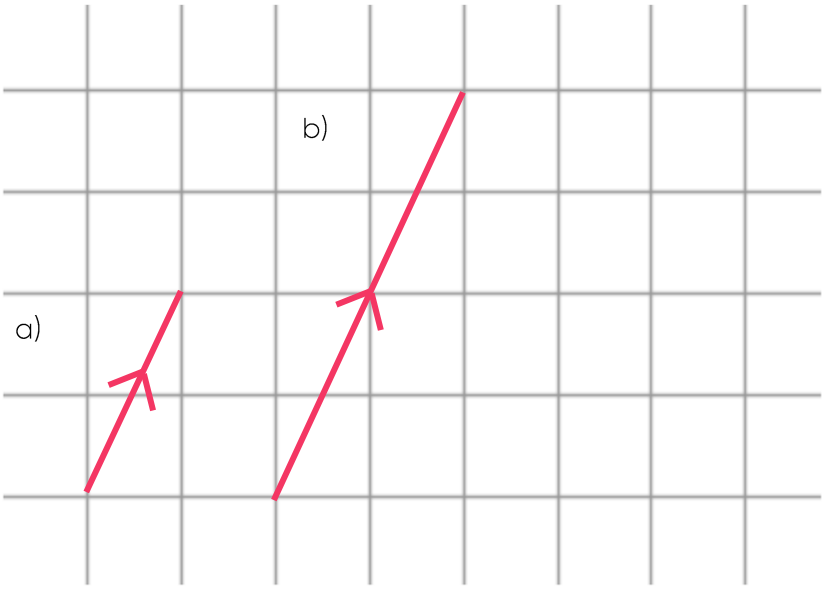


<u>Notes</u>

Part 3 - Scalar multipliers and basic addition of vectors

One of the attractive properties of vectors is that it is relatively easy to increase or decrease their length. The act of doing this is called scalar multiplication. Look at the grid below.

1. Write the 2 column vectors corresponding to each line. What do you notice?



Scalar multiplication of vectors follows intuitively. If we have a vector $\binom{3}{5}$, then 4 lots of this vector can easily be calculated as follows:

$$4\begin{pmatrix}3\\5\end{pmatrix} = 4 \times \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}4 \times 3\\4 \times 5\end{pmatrix} = \begin{pmatrix}12\\20\end{pmatrix}$$

You can see that multiplying a vector by a scalar simply requires multiplying each component by the scalar value.

1) $3\binom{2}{2}$

2) $6\binom{9}{27}$

3) $\frac{3}{2}\binom{4}{8}$

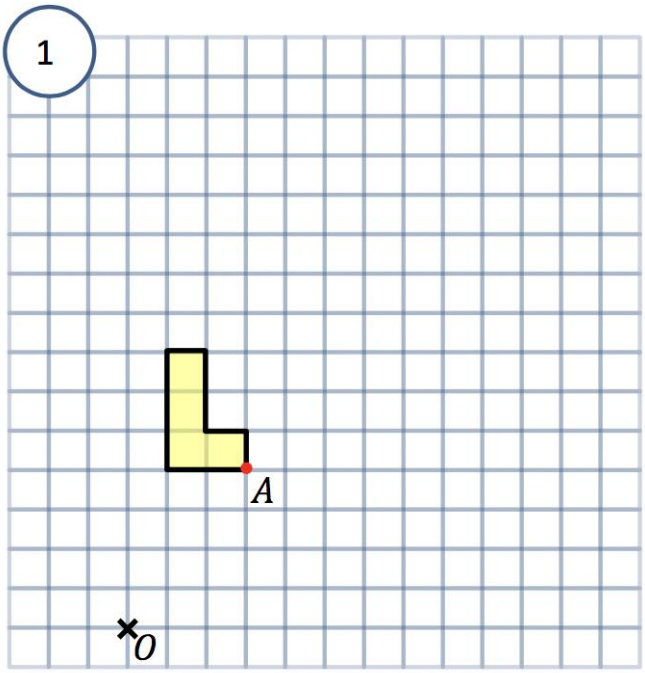
4) $3\binom{13}{-6}$

5) $-6\binom{13/4}{-6}$

Vectors can be used for enlargements as well as for translation. See if you can apply what you have learned about vectors and what you already know about enlargements in the following questions. Write down the vector that maps one original coordinate, labelled A, to the new coordinate A' using the scale factor given in each Question. The first has been given as a starting point

Worked example

1



This shape is to be enlarged by scale factor 2

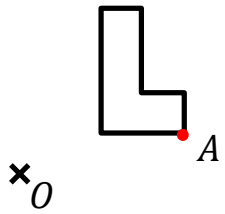
$\overrightarrow{\text{O to A}} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\overrightarrow{\text{O to A'}} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

Draw the image of the shape

Practice Question

2



This shape is to be enlarged by scale factor 3

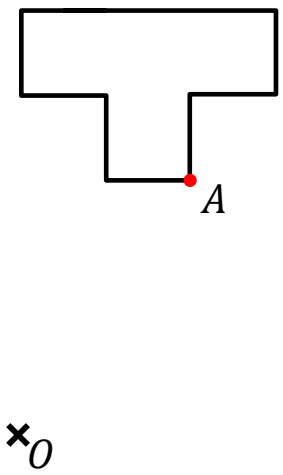
$\overrightarrow{OA} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

$\overrightarrow{OA'} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

Draw the image of the shape

Test yourself

3



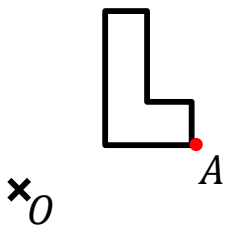
This shape is to be enlarged by scale factor $\frac{1}{2}$

$\overrightarrow{OA} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

$\overrightarrow{OA'} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

Draw the transformed shape

4



This shape is to be enlarged by scale factor -2

$\overrightarrow{OA} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

$\overrightarrow{OA'} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$

Draw the transformed shape

Tutorial 2: Vector notation

Objectives
1) How to label a vector
2) Vector arithmetic – combining vectors to make new vectors

Starter

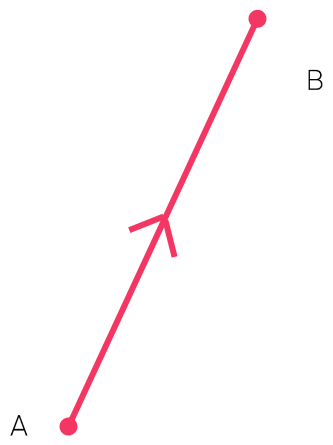
What vector takes me from the coordinate (4,5) to the coordinate (-3,4)?

Part 1 - How to label a vector

We have so far discussed the difference between vectors and scalars. It makes sense to have a way of labelling vectors such that we can immediately recognise we are talking about a vector quantity as opposed to a scalar quantity.

Worked example

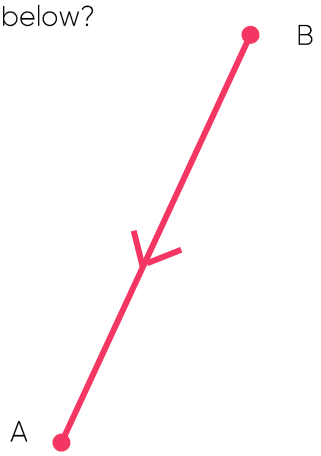
Look at the vector below…



This vector obviously represents a journey from A to B. The general convention is to label this journey as $\overrightarrow{AB}$, the arrow above the letters tells the reader that we are talking about a vector from A to B.

Practice Question

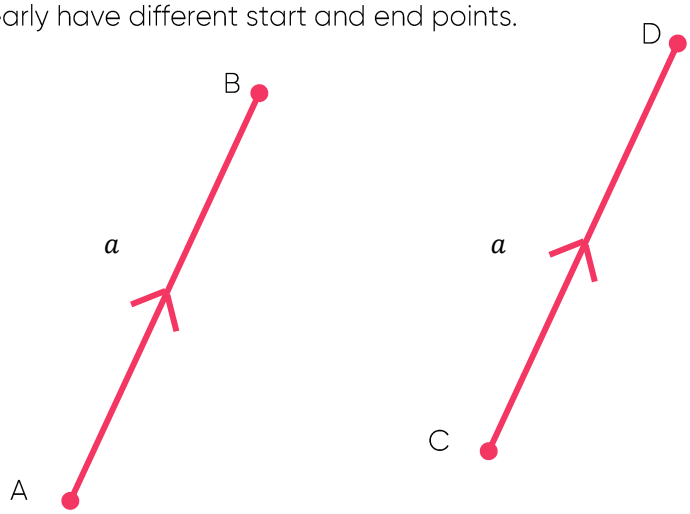
1) How might you label the vector below?



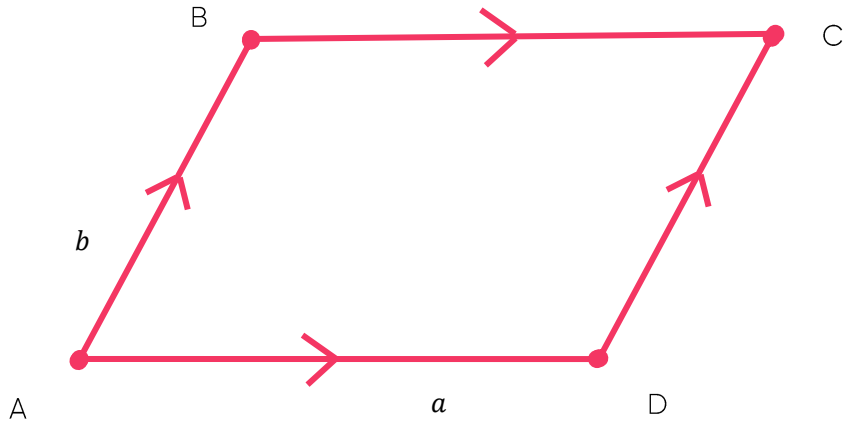
As was discussed in the previous tutorial a vector does not have any intrinsic information about a start point and end point, it just specifies a movement in a particular direction.

Worked example

The 2 vectors below have the same magnitude (length), and represent a journey in exactly the same direction, BUT clearly have different start and end points.



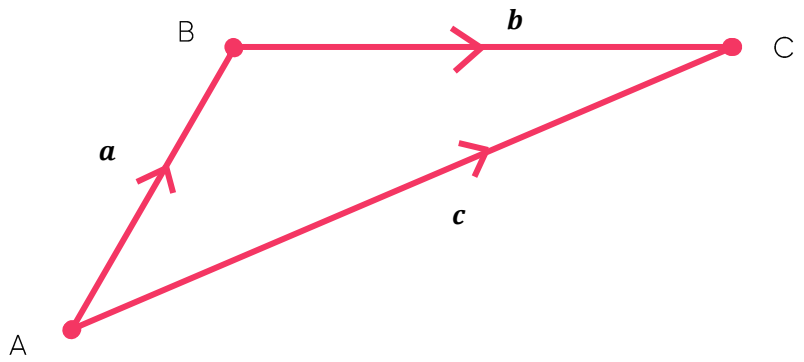
These journeys are represented by the same vector, here we have called this vector $\boldsymbol{a}$. The bold indicates that this is a vector and not a scalar!

Practice Question
2) ABCD is a parallelogram, write $\overrightarrow{BC}$ and $\overrightarrow{CD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. 

Part 2 – Vector arithmetic

So far, we've looked at how vectors can be represented visually, in column format and using vector notation. We've also learned how to multiply vectors by scalar quantities, changing their magnitude but not their direction. Conveniently, we can also add vectors together to make new vectors (we've actually already seen this above in the drawing of shapes!).

If I start by walking from A to B and then to C in the example below, where $\overrightarrow{AB}$ is given by the vector $\boldsymbol{a}$ and $\overrightarrow{BC}$ is given by the vector $\boldsymbol{b}$, we form a new single journey, $\overrightarrow{AC}$ given by $\boldsymbol{c}$.



By adding the two vectors together we make a new vector, and we can write this in an easy to read format:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$$

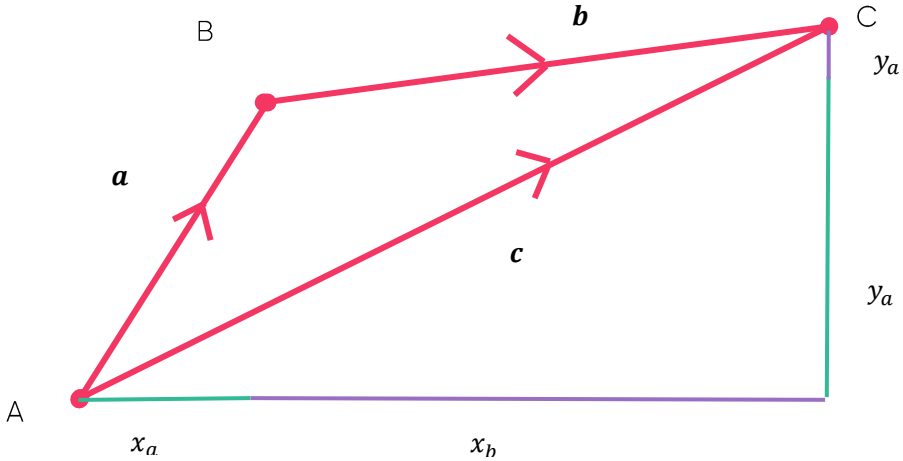
Now you can see vector addition represented pictorially and how we show this in compact vector notation. Addition can also be extended to column vectors. Keeping things general, if we have a column vector, $\boldsymbol{a} = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} x_b \\ y_b \end{pmatrix}$, we can add these two vectors together by first adding the x components (top numbers) and separately adding the y components

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} \boldsymbol{x_a} \\ \boldsymbol{y_a} \end{pmatrix} + \begin{pmatrix} \boldsymbol{x_b} \\ \boldsymbol{y_b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x_a + x_b} \\ \boldsymbol{y_a + y_b} \end{pmatrix}$$

(bottom numbers) as follows:

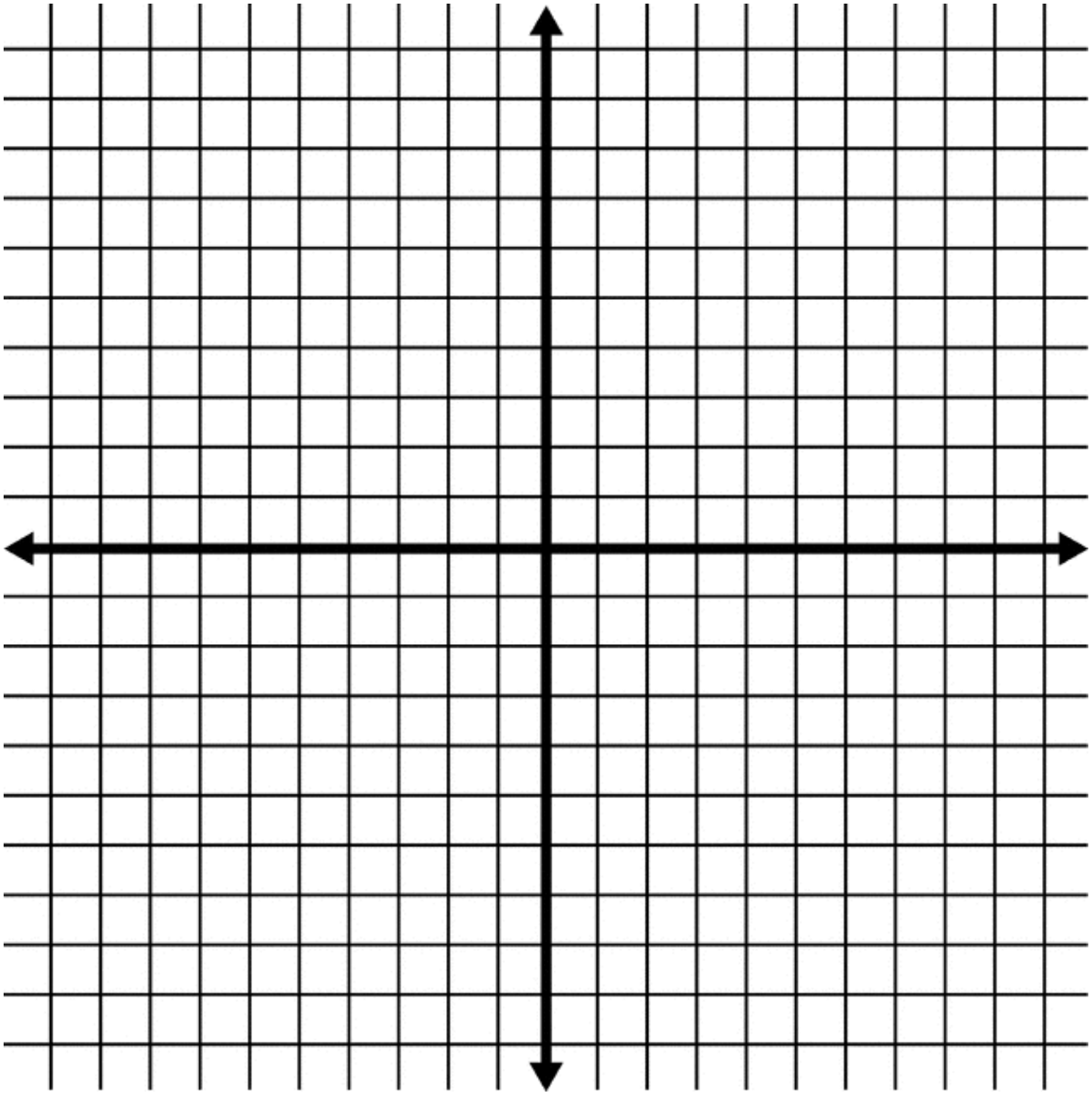
If we look back to the drawing of these vectors, we can see how this works.

Given that we can add vectors, analogously to scalars, we can also subtract them. Instead of adding the x,y-components we take them away.



Practice question
Find the resulting column vector when two vectors, $\boldsymbol{a} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ are (a) added together and (b) one subtracted from the other. (Be careful here, this should give you two values, are they the same? What does this tell you about vector subtraction?).

Practice Question
Discuss: Are the vectors in the previous question parallel? How do you know
If point *g* is at coordinate (5,4) and point *h* is at coordinate (3, 1), write the column vector that describes the journey from *g* to *h*? (hint: a diagram may help!)



Test yourself

1: Calculate the following:

a) $\begin{pmatrix} 0 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix}$ b) $\begin{pmatrix} -9 \\ -8 \end{pmatrix} + \begin{pmatrix} -10 \\ -2 \end{pmatrix}$ c) $\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -5 \\ 8 \end{pmatrix}$ d) $\begin{pmatrix} -8 \\ -4 \end{pmatrix} + \begin{pmatrix} -6 \\ -5 \end{pmatrix}$

_____ _____ _____ _____

2: Calculate the following:

a) $\begin{pmatrix} -2 \\ -4 \end{pmatrix} - \begin{pmatrix} -3 \\ -7 \end{pmatrix}$ b) $\begin{pmatrix} -10 \\ 8 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix}$ d) $\begin{pmatrix} -2 \\ -9 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

_____ _____ _____ _____

3: Calculate the following:

a) $0\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ b) $6\begin{pmatrix} 10 \\ 0 \end{pmatrix}$ c) $6\begin{pmatrix} 10 \\ 10 \end{pmatrix}$ d) $-2\begin{pmatrix} 6 \\ -3 \end{pmatrix}$

_____ _____ _____ _____

4: Calculate the following:

a) Are vectors $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -20 \\ 35 \end{pmatrix}$ parallel?

_____

b) Are vectors $\begin{pmatrix} 32 \\ -16 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ parallel?

_____

c) Are vectors $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ parallel?

_____

d) Are vectors $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -30 \\ -3 \end{pmatrix}$ parallel?

_____

5: Calculate the following:

a) If A = (- 3, - 1) and B = (9, 3), find the vector from A to B.

_____

b) If A = (9, 4) and B = (2, - 4), find the vector from A to B.

_____

c) If A = (2, - 9) and B = (4, 1), find the vector from B to A.

_____

d) If A = (6, - 7) and B = (- 2, - 5), find the vector from B to A.

_____

Tutorial 3 – Vectors in geometry

Objectives
1) Invariance of paths – loops, and a change of direction
2) Collinearity – taking the idea of parallel vectors further to prove if 3 points form a straight line

Starter

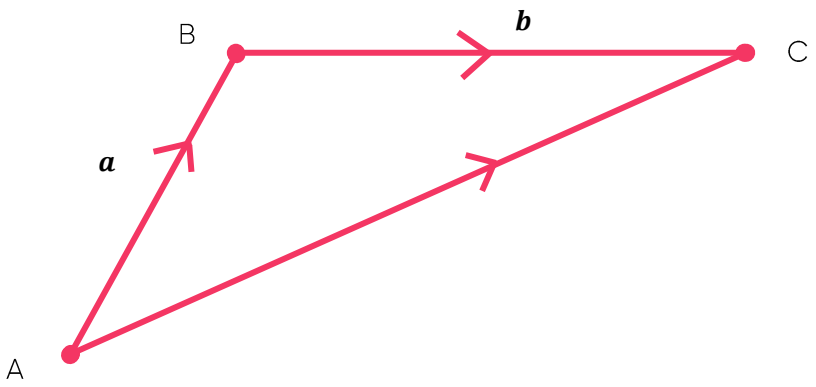
I have 3 vectors and I am told that 2 of them are parallel. What is the minimum number of comparisons I must make between the vectors to check which two are parallel?

Part 1 – Invariance of paths

As we have already seen, vectors are great for drawing geometric shapes. It is, however, really important to keep track of which vector represents which specific journey, remembering that the direction is crucial!

Worked example

Let's look at a previous example in a slightly different way. We want to write $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

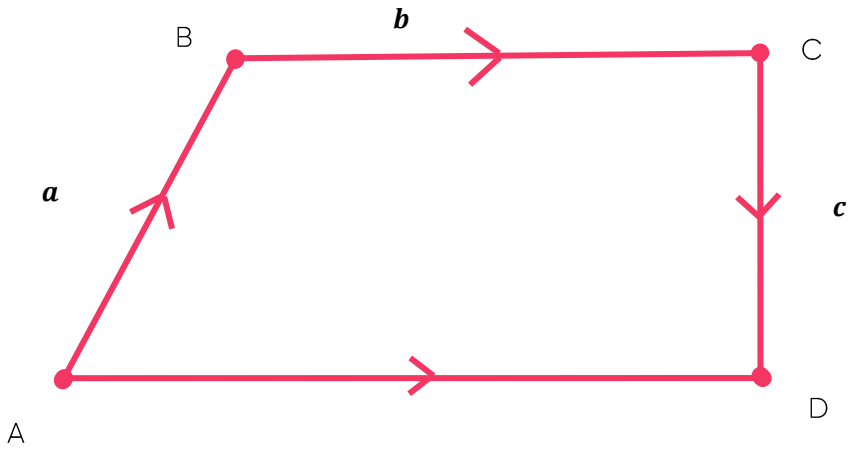


We saw in the previous tutorial that we can add vectors together, but we didn't really consider what this means in reality. You can see that if we start at point A and walk along the path $\overrightarrow{AB}$, then along the path $\overrightarrow{BC}$, we arrive at point C. The net result would be the same if we walk directly along the path $\overrightarrow{AC}$. This is known as the invariance of paths. When using vectors, the path taken does not matter, only the beginning and end point count. Mathematically, this is just addition of vectors from last time…

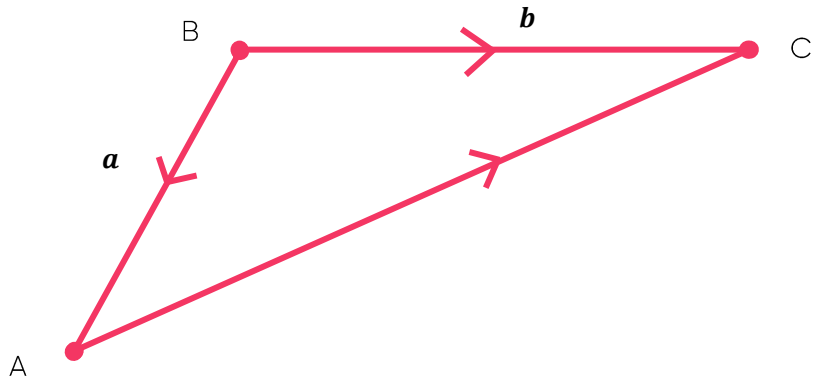
$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$$

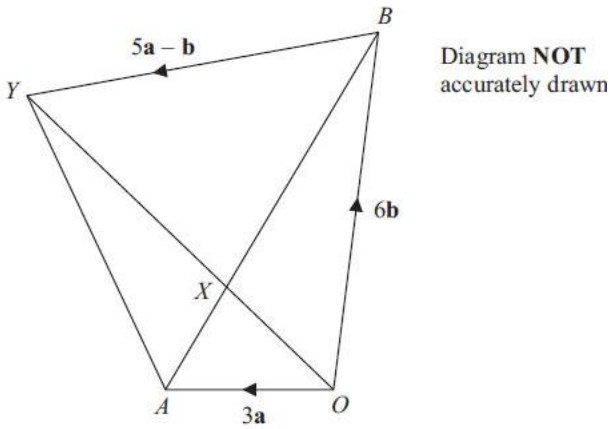
Practice Questions

ABCD is a right-angled trapezium, write $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



Write $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.





OAYB is a quadrilateral.

$\overrightarrow{OA} = 3\mathbf{a}$

$\overrightarrow{OB} = 6\mathbf{b}$

(a) Express $\overrightarrow{AB}$ in terms of a and b.

.....

(1)

(b) Express $\overrightarrow{AY}$ in terms of a and b.

.....

(2)

Express $\overrightarrow{OY}$ in terms of a and b.

.....

(2)

Test yourself!

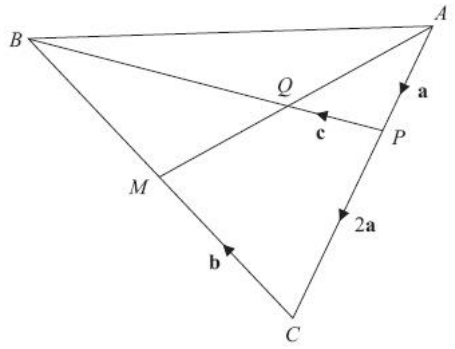


Diagram **NOT** accurately drawn

M is the midpoint of BC.
Q is the midpoint of AM.

$\overrightarrow{AP} = \mathbf{a}$ $\overrightarrow{PC} = 2\mathbf{a}$ $\overrightarrow{CM} = \mathbf{b}$ $\overrightarrow{PQ} = \mathbf{c}$

(a) $\overrightarrow{AM}$ Find in terms of a and b.

$\overrightarrow{AM}$ = ...

(1)

(b) Find $\overrightarrow{QB}$ in terms of c. (CHALLENGE)

Hint: Find in terms of a and b first.

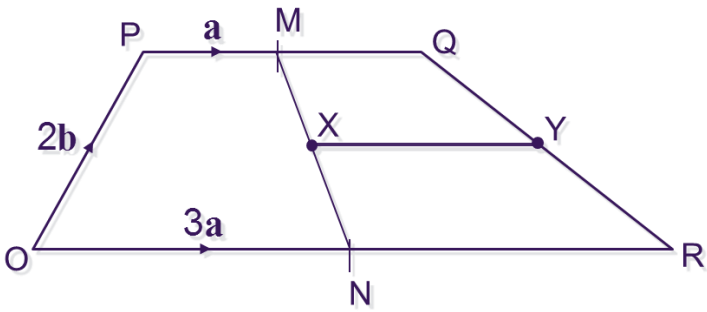
$\overrightarrow{QB}$ = ...

(4)

Part 2 – Collinearity

If three or more points are collinear, they all lie on a straight line. Examiners will often ask vector questions that require you to show whether a set of points lie on a straight line. In effect, you are actually testing whether two (or more) vectors are parallel, which we have already looked at. Here are some, potentially tricky, examples of such questions…

Worked example



X* is the midpoint of *MN
Y* is the midpoint of *QR.

Find the vector MN in terms of a and b.

Prove that *XY* is parallel to *OR.*

Practice Question

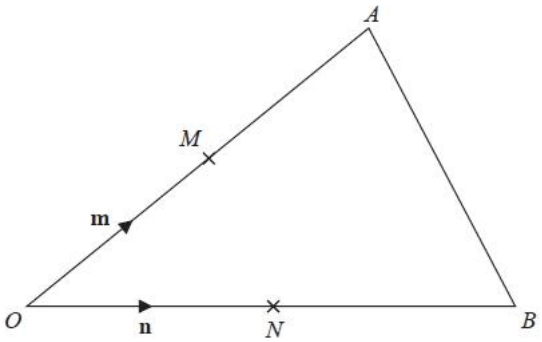


Diagram **NOT** accurately drawn

OAB is a triangle.
M is the midpoint of OA.
N is the midpoint of OB.

$\overrightarrow{OM}$ = m

$\overrightarrow{ON}$ = n

Show that AB is parallel to MN.

Test yourself

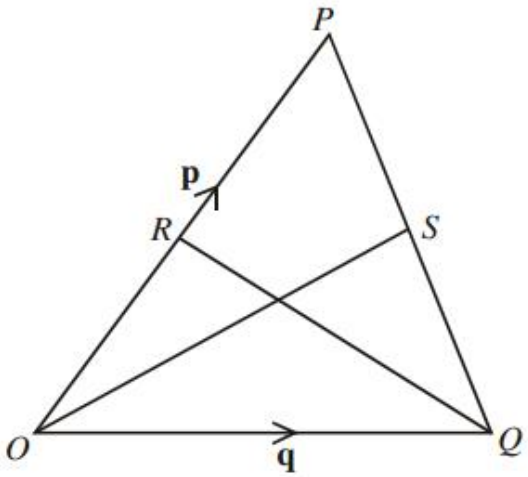


Diagram **NOT** accurately drawn

OPQ is a triangle.

R is the midpoint of OP.

S is the midpoint of PQ.

$\overrightarrow{OP} = p$ and $\overrightarrow{OQ} = q$

(i) Find $\overrightarrow{OS}$ in terms of p and q.

$\overrightarrow{OS}$ = ...

(ii) Show that RS is parallel to OQ.

(Total 3 marks)

Tutorial 4 – Synoptic questions

Objectives
1) Combining vectors with ratio
2) Vectors and trigonometry

As with most of mathematics, the difficult questions are those that require you to recall knowledge from several different topics, these questions are known as synoptic questions. Let's look back at a slightly changed question from last week...

Part 1 – Combining vectors with ratio

Worked example

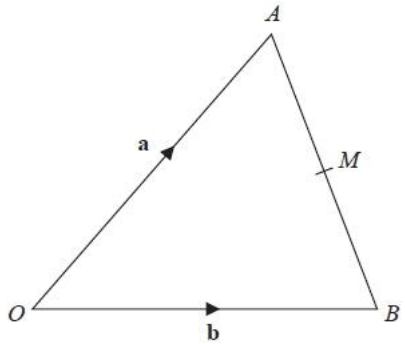


Diagram **NOT** accurately drawn

OAB is a triangle.

$\overrightarrow{OA} = \mathbf{a}$

$\overrightarrow{OB} = \mathbf{b}$

M is the midpoint of *AB*.
OMN is a straight line such that *ON* : *OM* = 3 : 2

Find, in terms of **a** and **b**, an expression for the vector $\overrightarrow{ON}$.
Write your answer in its simplest form.

Practice question

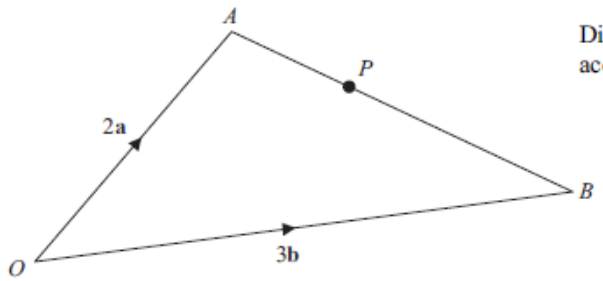


Diagram **NOT** accurately drawn

OAB is a triangle.

$\overrightarrow{OA} = 2\mathbf{a}$

$\overrightarrow{OB} = 3\mathbf{b}$

(*a*) Find AB in terms of **a** and **b**.

$\overrightarrow{AB}$ = ...

(1)

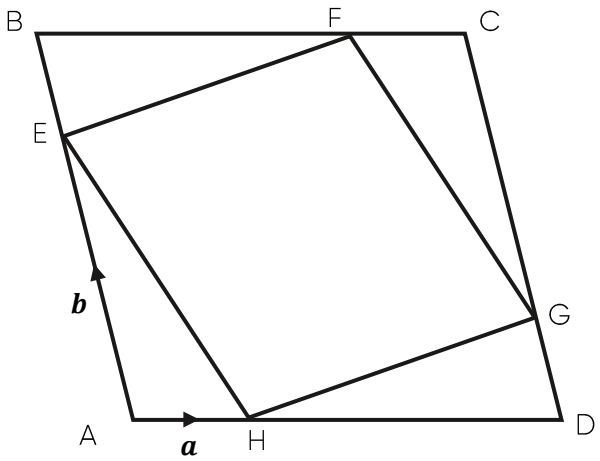
P is the point on AB such that $AP : PB = 2 : 3$

(*b*) Show that $\overrightarrow{OP}$ is parallel to the vector **a** + **b**.

Test yourself

In the diagram below ABCD is a paralellogram.

The ratios AE:EB, BF:FC, CG:GD and DG:HA are all 2:1.



$\overrightarrow{AH} = \boldsymbol{a}$ and $\overrightarrow{AE} = \boldsymbol{b}$. Show that EFGH is also a paralellogram.

Challenge – About as hard as it can get for vectors at GCSE!

(16) Two vectors are defined as follows:

$$\overrightarrow{AC} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

Find the value of $\cos(ACB)$ in its simplest form.

Feedback

What is the Purpose of the feedback tutorial?

- To receive feedback on final assignments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.
- To reflect on the programme including what was enjoyed and what was challenging.

Final assessment feedback

What I did well...	What I could have improved on...
•	•
•	•
•	•

My target for future work is...

Reflecting on Subject Enhancement

What did you most enjoy about the Subject Enhancement course?
•   •   •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

Notes

Notes

Notes

Vectors

Key Stage 4 Programme

researchersinschools.org