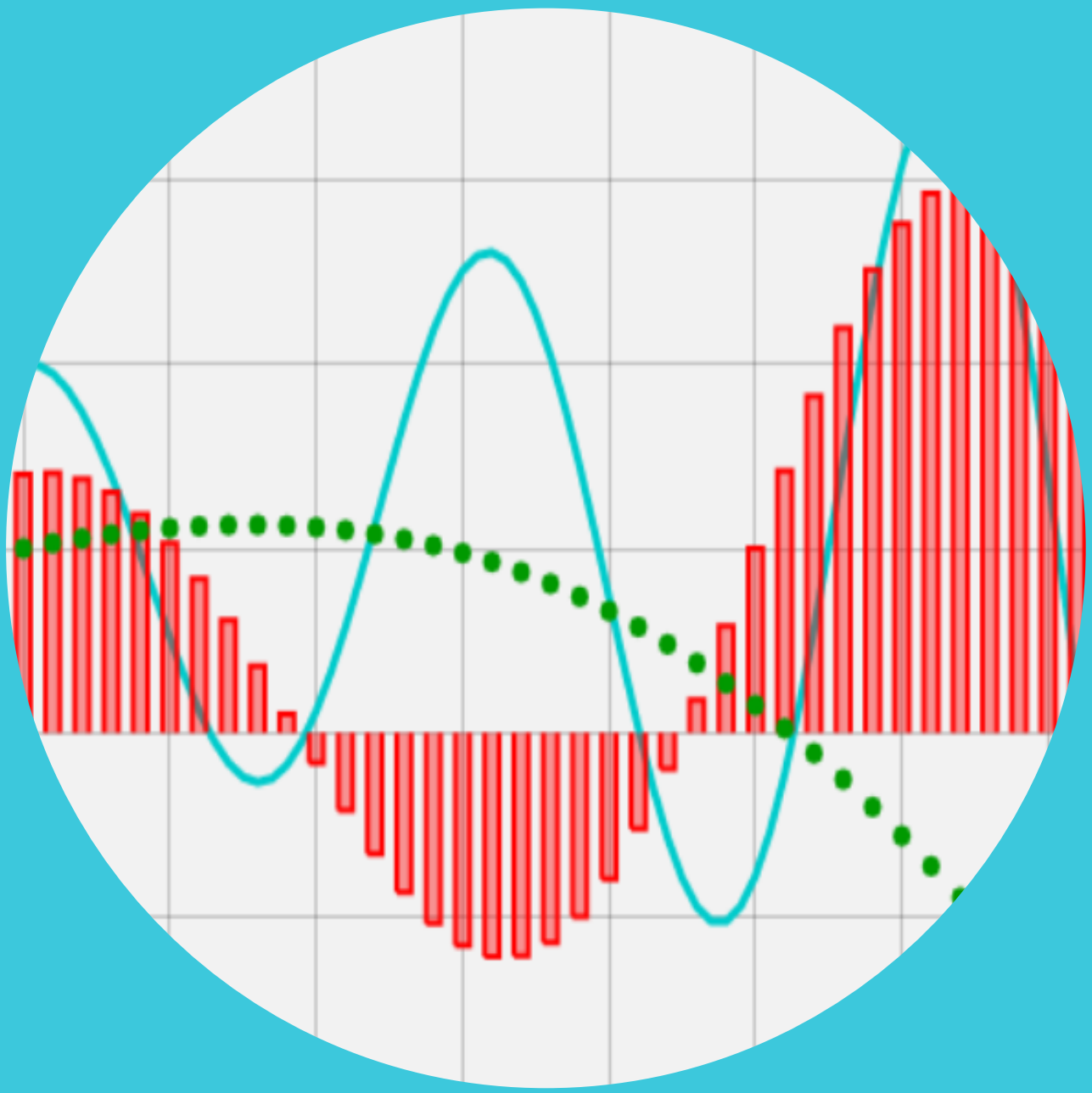
Researchers in Schools



# Graphs – The subtle lines between Maths and real life

## Key Stage 4 Programme

| | |
|---|---|
| Pupil Name | |
| Handbook Designed by | Dr Pietro Liuzzo Scorpo |

## Timetable – Tutorials

| Tutorial | Date | Time | Location |
|---|---|---|---|
| 1 (Baseline assessment) | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 (Final assessment) | | | |
| 7 (Feedback) | | | |

## Timetable – Homework Assignments

| Homework Assignment | Description | Due Date |
|---|---|---|
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| Tutorial 2 | | |
| Tutorial 3 | | |
| Tutorial 4 | | |
| Tutorial 5 | | |

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# Course Rationale

Mathematics is a multifaceted language capable of describing with a multitude of representations the world surrounding us. Many of these representations are not immediately accessible: formulae and equation are probably the most paradigmatic example – even though they contain all the information one could need, this information is hindered by letters, numbers and symbols that require a translation of some kind to be really understood.

In this year's Uni Pathways course, we are going to investigate the most common way to represent a mathematical relationship: graphs. Indeed, graphs are widely used to convey important information not only between scientists and mathematicians, but also (and in particular) to and between the general public. Graphs and charts appear daily in the news, during meetings in the workplace, when analysing the success of a social media account, in memes... This widespread, ubiquitous and crucial role in our society is reflected in the importance given to them in the Maths curriculum and their certain presence in the GCSE Maths exam.

During the series of this year's tutorials we will revise how to draw, interpret and extract both qualitative and quantitative information from graphs.

The goal of this year's Uni Pathways is not solely that of revising topics that you have encountered already, but it is also that of learning how to revise. Several studies highlighted that many students revise by rereading text and making notes. These kind of strategies are not very effective: do you think you get better at football, or at playing an instrument, by simply reading a text about football techniques or about playing the instrument? Of course not! It is obvious that to get better at these things, you need to practice! For a school subject, and in particular for Maths, is no different: you need to practice, expose yourself to many different and varied examples as possible. This is what we are going to do in these tutorials: revision by deliberate practice!

# Subject Vocabulary

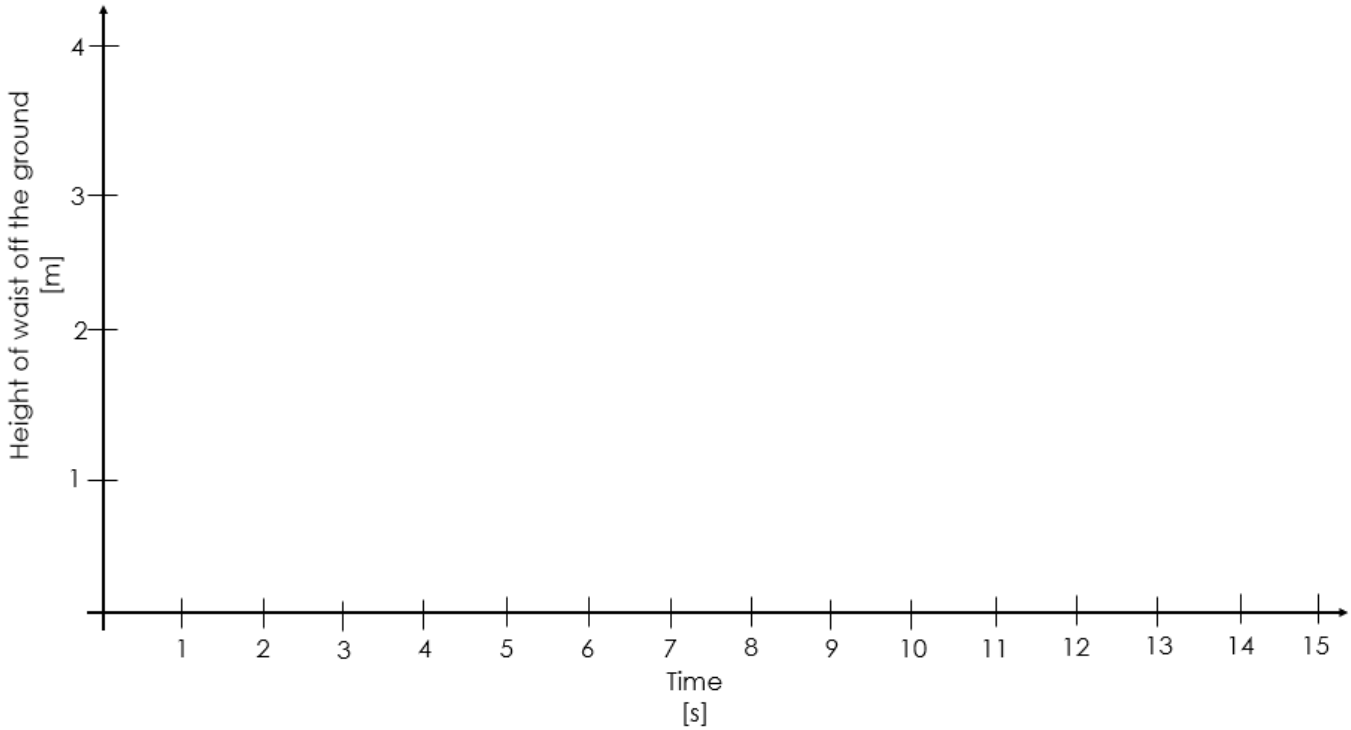
| Word | Definition | In a sentence |
|---|---|---|
| Graph | Pictorial representation of the relationship between two quantities (usually called x and y) | The graph represents the distance travelled by a train between 1pm and 3:30pm. |
| Gradient (slope) | The gradient of a straight line is a quantity that tells us how steep a straight line is. | A horizontal straight line has a 0 gradient. |
| Rate of change | This is a quantity (or a ratio) which tells us how much one quantity changes when another quantity changes by 1. | The rate of change between the British pound and the Euro is 0.89£/€<br>£0.89 : 1€ |
| Average rate of change between two points | It is the gradient of the line connecting two points on a graph. | - |
| Instantaneous rate of change (at a point) | It is the gradient of the line tangent to the graph in a given point of a graph. | - |

# Tutorial 1 – The story behind the line

| Objectives |
| --- |
| A. Interpreting graphs – Narrating what happens by looking at a graph.<br>B. Determining and interpreting the rate of change of linear graphs. |

## A. Interpreting graphs

| Knowledge |
| --- |
| Graphs are a common way to represents data and/or a relation between two quantities. These two quantities are represented on the two axis. Always remember to label your axis and to write the units used to measure the quantities.<br><br>The graphs we are interested in are lines and curves that tells us what the value of the quantity on the y-axis is when we know the value of the quantity on the x-axis. In science you might have called them *dependent* (y-quantity) and *independent* (x-quantity) *variables,* respectively: the dependent variable is the one that we measure when the independent variable changes.<br><br>When we are presented with a graph, we should<br>1. Identify the quantities involved and their units<br>2. "Read" the line (or curve): 'When x is …, *y* is …'<br><br>**Example 1**<br><br>Watch the video and draw the graph. |

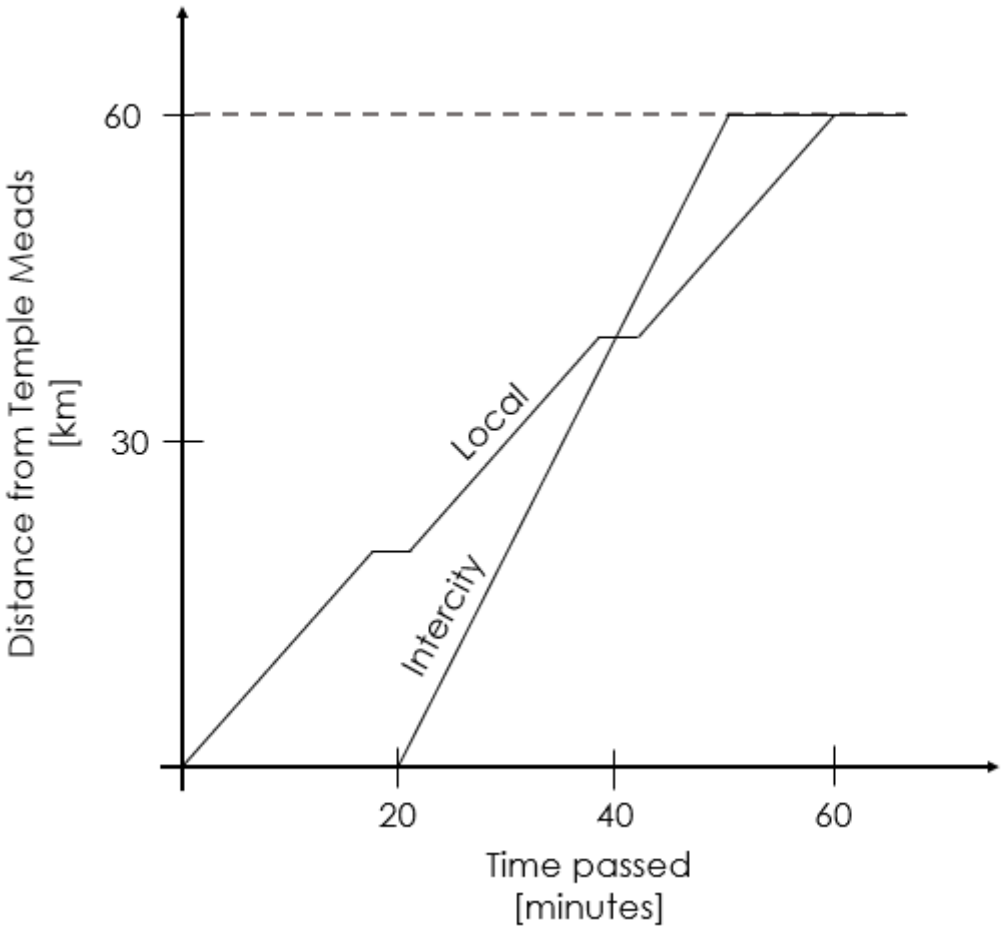



On your graph, for each part of the curve, describe what happens in the video.

**Example 2**

A local train leaves Bristol Temple Meads at 3:17pm.

a) At what time does the Intercity train leave Bristol Temple Meads?
b) How long does it take to the Intercity train to reach its destination, 60km from Bristol?
c) At what time does the local train stop for the second time?



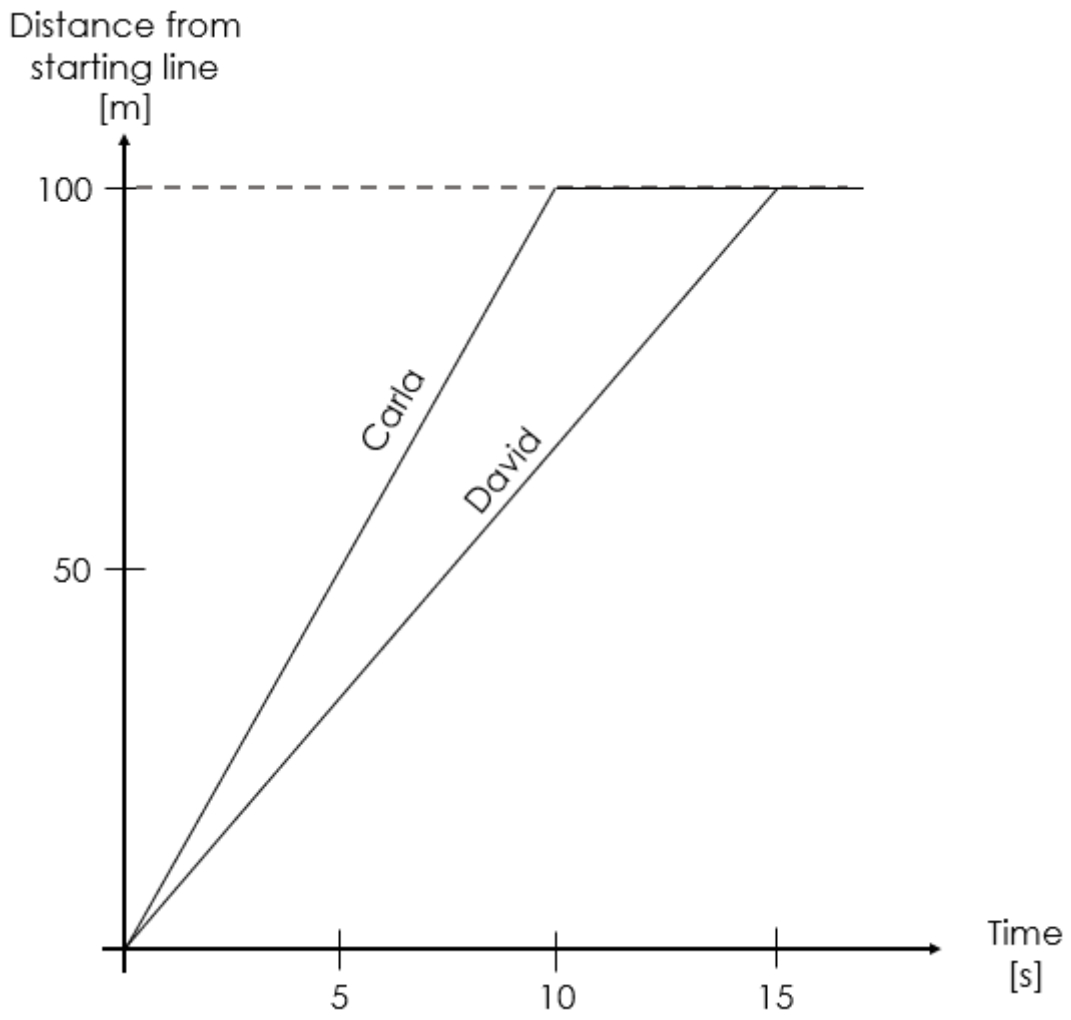

The y-quantity (dependent variable) is the distance (measured in km) from Bristol Temple Meads. The x-quantity (independent variable) represents time (measured in minutes) passed from a certain instant.
At the initial time the Local train is at a distance 0km from the station: this means that the train is in Bristol Temple Meads. As time passes this train gets further away from Bristol.
Since we know that the Local train leaves at 3.17pm we can say that the x-quantity represents the minutes passed from 3.17pm. We can therefore answer the first question:
*The intercity train leaves 20 minutes after 3.17pm:* ***3.37pm***.
The intercity leaves and, as time passes, it gets further from Bristol. It is at a distance of 60km from Bristol (its destination) 50 minutes (we read on the x-axis) after the initial time, 3.17pm. The time it took is easily found:
*Arrival time – Departure time = 50min – 20min =* ***30 minutes***.
To answer the last question, we have to think how the train stopping is represented: when the train stops it does not get further from the station, which means that even though time is passing the distance does not increase. The stop is therefore represented by a line parallel to the x-axis.
*The second stop happens at about 40 minutes after the departure:* **3:57pm.**

Practice

**Exercise 1**

The graph represents a 100m race between Carla and David.

a) Who wins the race?
b) What is their time?
c) How much longer does it take to the second one to complete the race?

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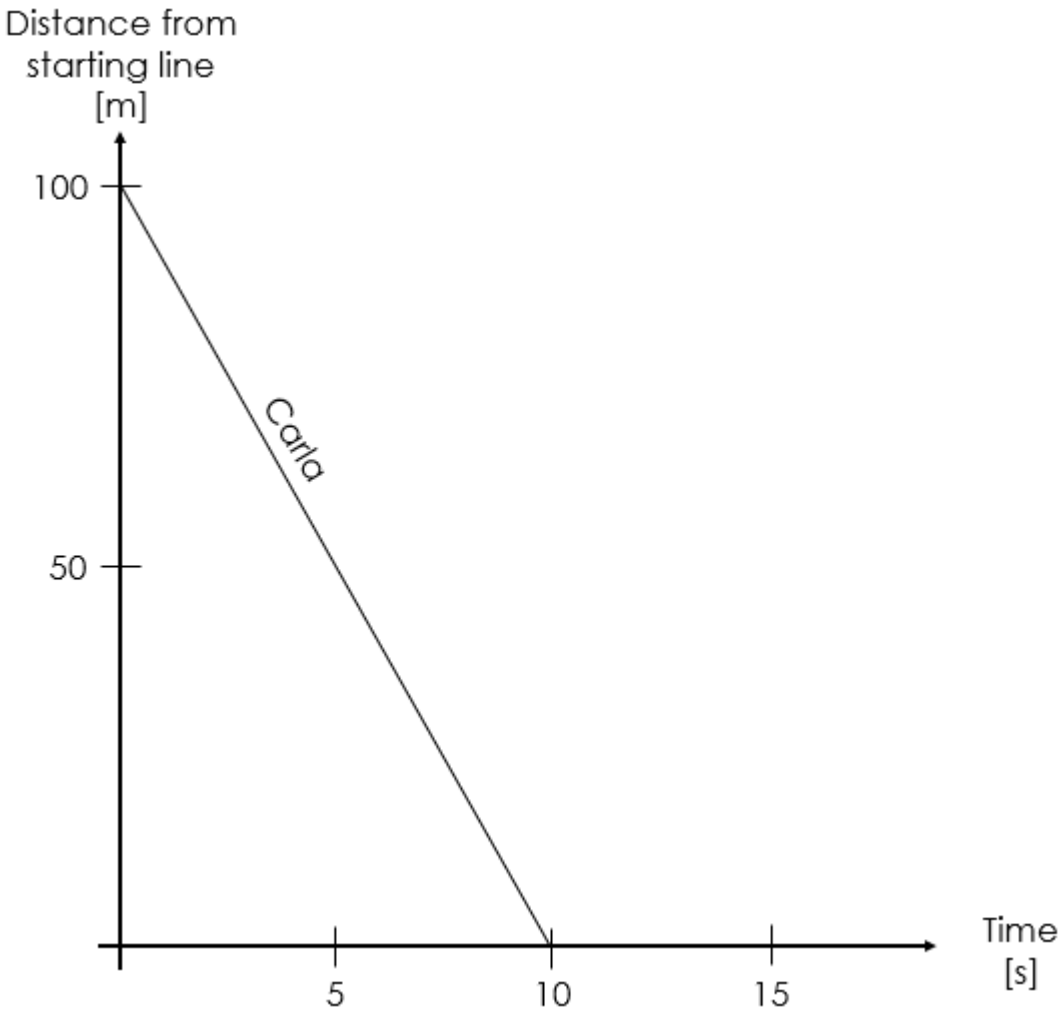
1. a) Carla b) 10s c) 5s

**Exercise 2**

The graph represents Carla running on a 100m race track.

a) Where is Carla at the initial time (0s)?
b) What is the distance she runs in 5s?
c) Where is Carla after 10s?

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1. a) On the finish line b) 50m c) On the starting line

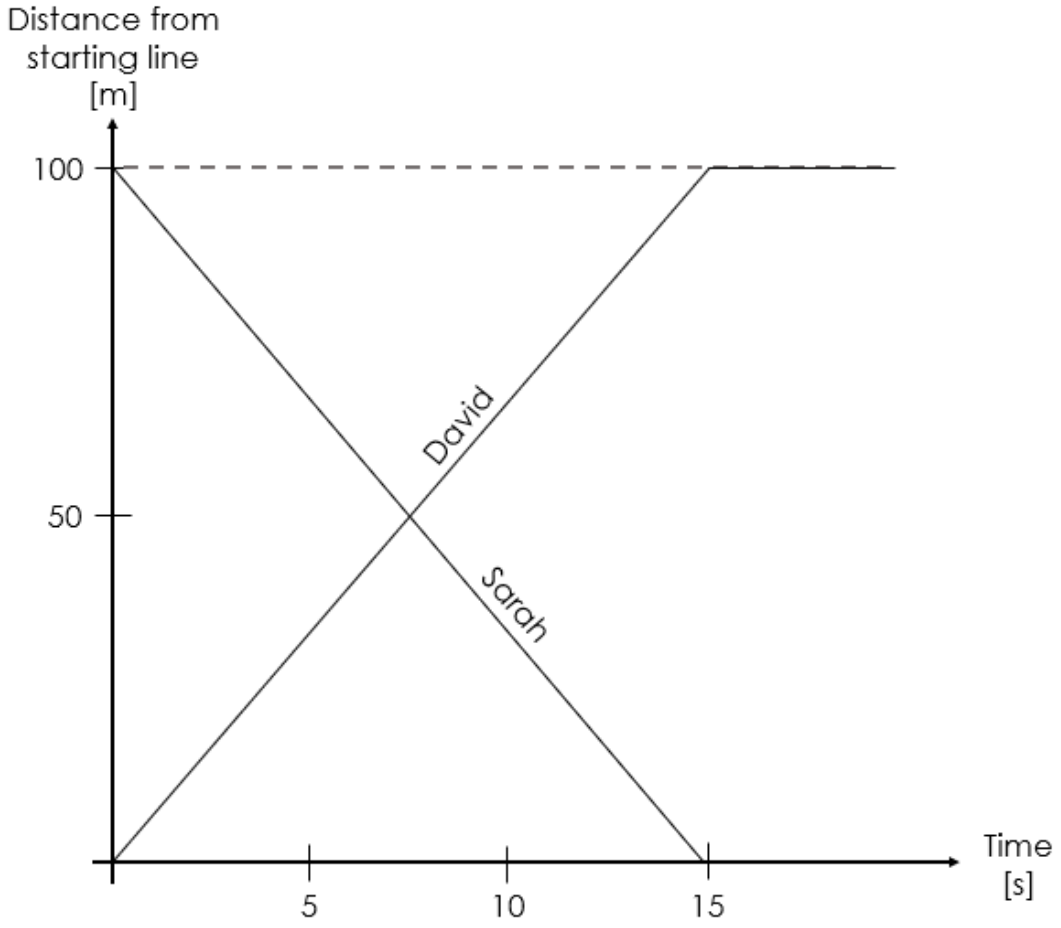
**Exercise 3**

The graph represents David and Sarah running on a 100m race track.

a) How long does it take David to run 100m?
b) Who is faster: Sarah or David?
c) What happens at 7.5s?

George starts his run with Sarah. He follows her at a constant, but slower, speed: it takes him 10 seconds to run the first 50m.

d) Draw George's graph.

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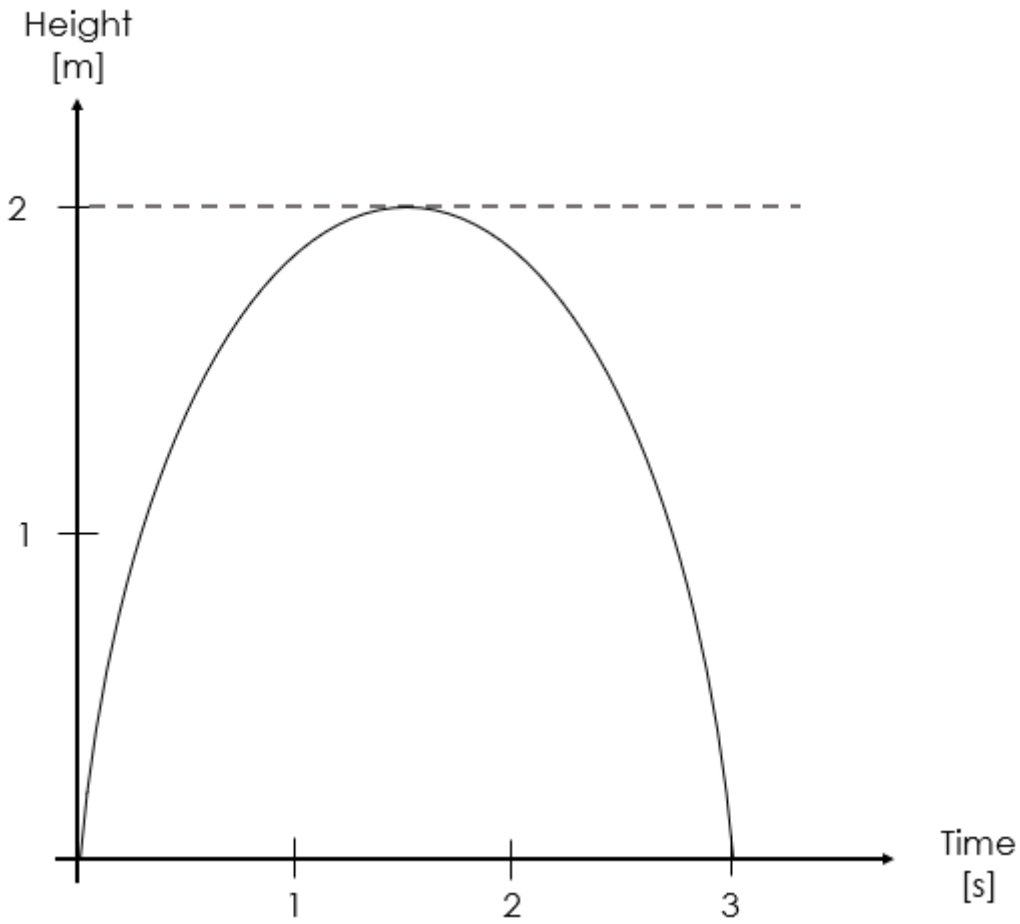
1. a) 15s b) They run at the same speed c) They cross each other in the middle of the race track (at 50m)

**Exercise 4**

The graph describes the trajectory of a tennis ball being thrown up vertically.

a) After how many seconds does the tennis ball reach the highest point?

b) How much distance has the ball travelled after 3 seconds?

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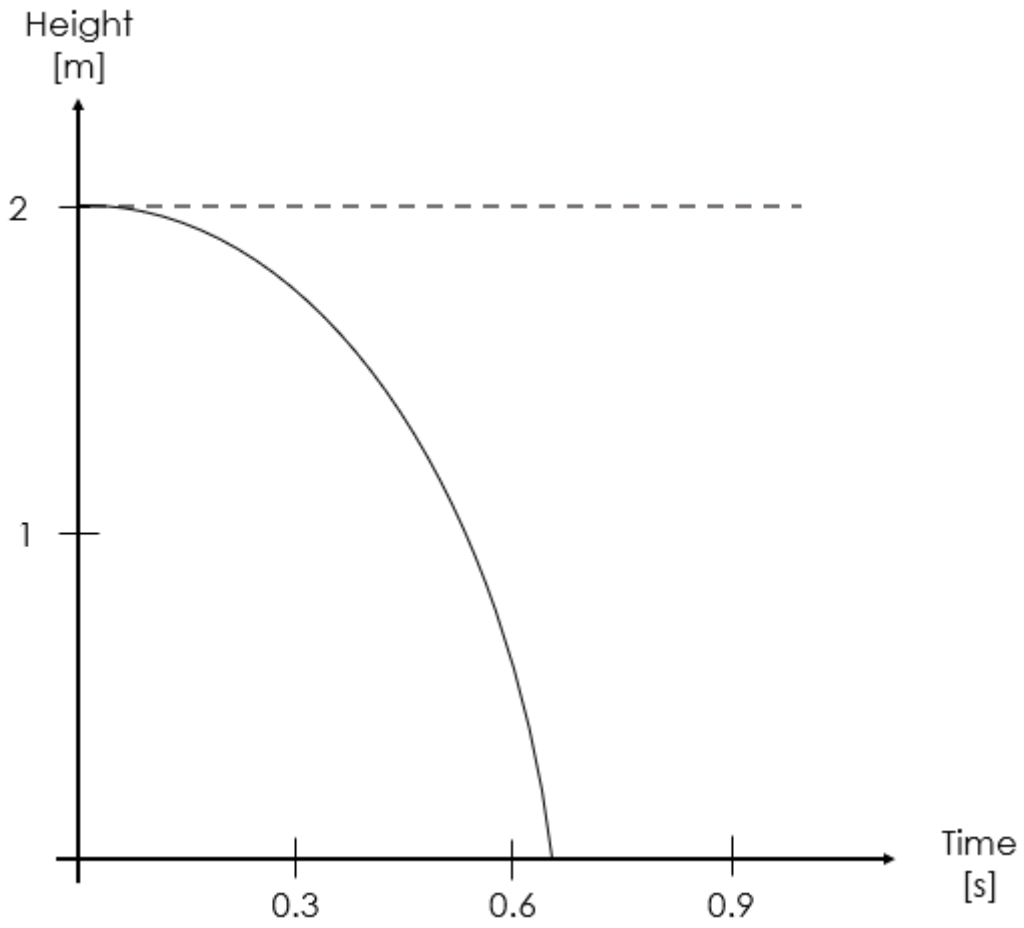
1. a) 1.5s b) 4m: 2m up and 2m down

**Exercise 5**

The graph describes the free fall of an object.

a) How much time does it take the object to reach the ground?
b) How far did the object fall after 0.3s?
c) How high the object is after 0.6s?

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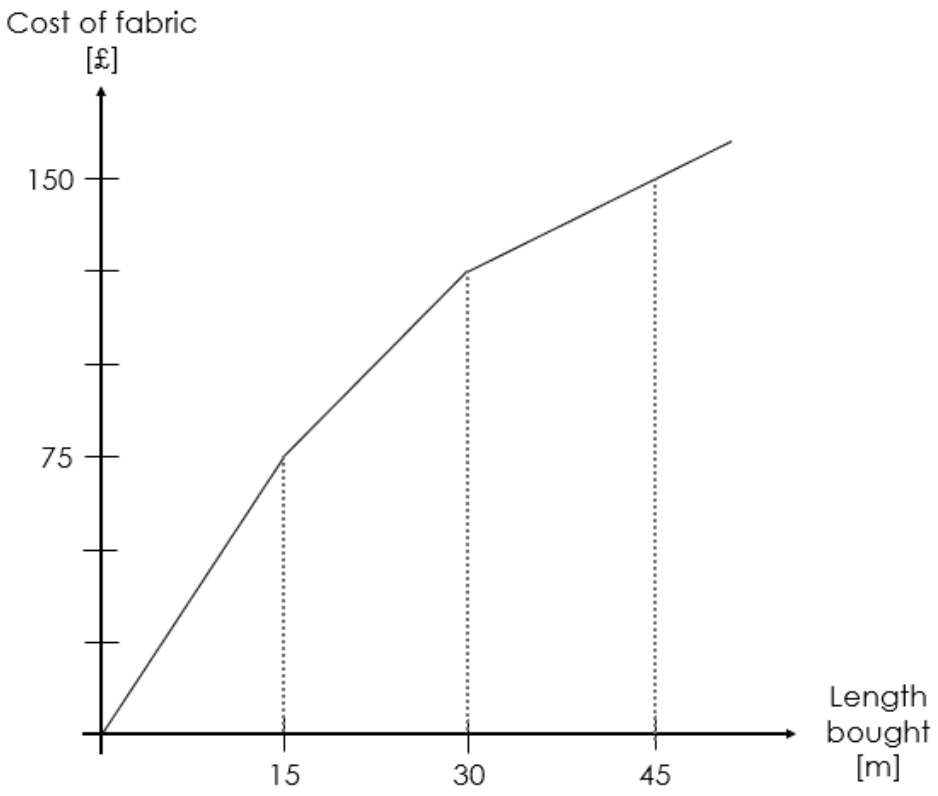
1. a) 0.67s b) 0.3m c) 0.6m

**Exercise 6**

The cost of fabric is represented in the graph.

a) How much does it cost to buy 15m of fabric?
b) If you buy 30m of fabric, how much do you spend per metre?
c) How much does it cost to buy 30m of fabric?
d) How much does it cost to buy 45m of fabric?
e) How much does every metre bought **after 30m** cost?

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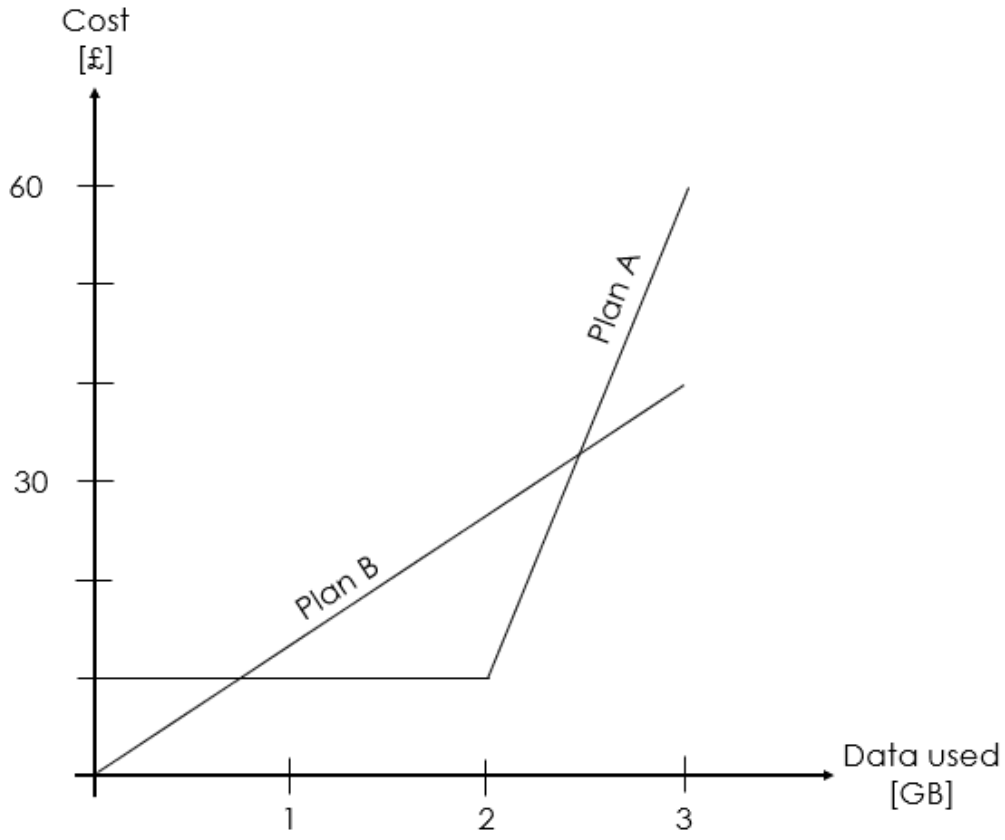
1. a) £75 b) £5 c) £125 d) £150 e) £1.67

**Exercise 7**

Two data monthly plan costs are represented in the graph.
Mark estimated that he uses between 1.5 and 1.8 GB of data every month.
Sheila estimated that she uses almost 3GB of data every month,
Carl estimated that he uses less than 1GB of data every month.
What plan should each of them choose? Justify your answer.

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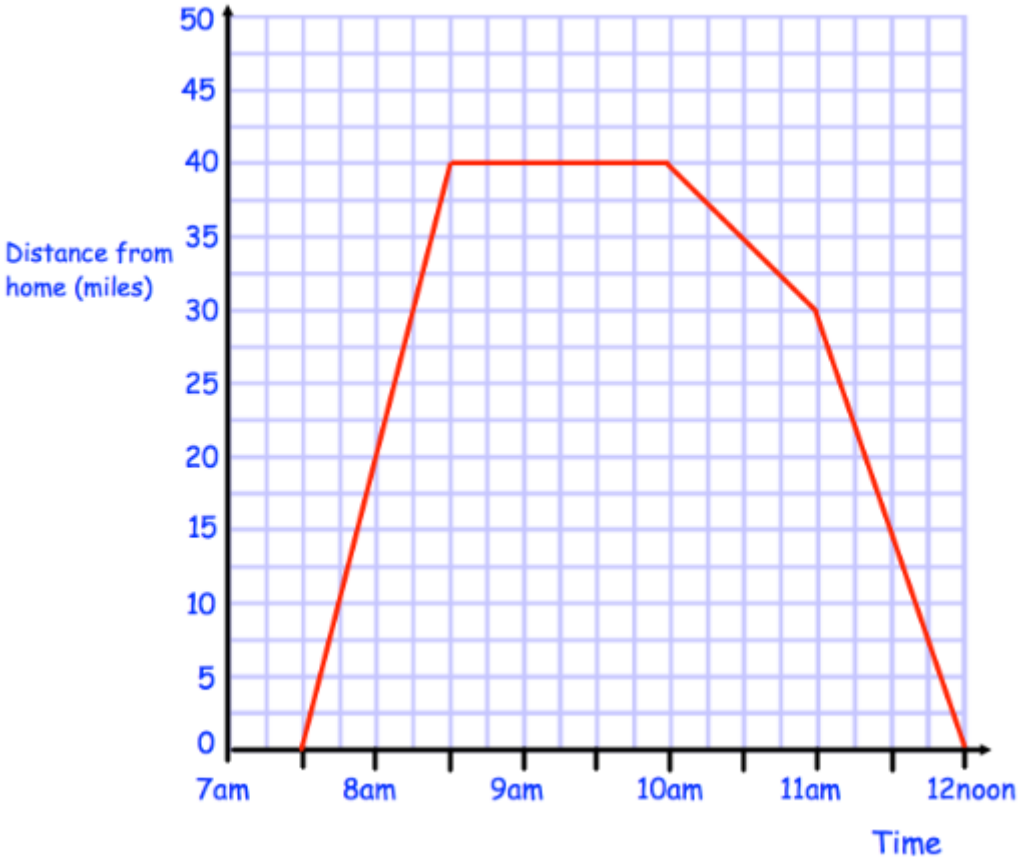
Mark: plan A, Sheila: plan B, Carl: it depends

Assessment

**Question 1** *(question adapted from CorbettMaths)*

Emma travelled to her Grandmother's house and back.
The distance-time graph shows information about her journey.




a) What time did Emma begin her journey?

7am 7.15am 7.30am 7.45am

b) How far was Emma from home at 8am?

17.5miles 20miles 15miles none of the previous

c) How long did Emma stay at her Grandmother's house?

10am 1h 30' 2h 8.30am

d) What time did Emma leave her Grandmother's house?

11am 10.30am noon none of the previous

e) How far was Emma from her Grandmother's home at 11:45?

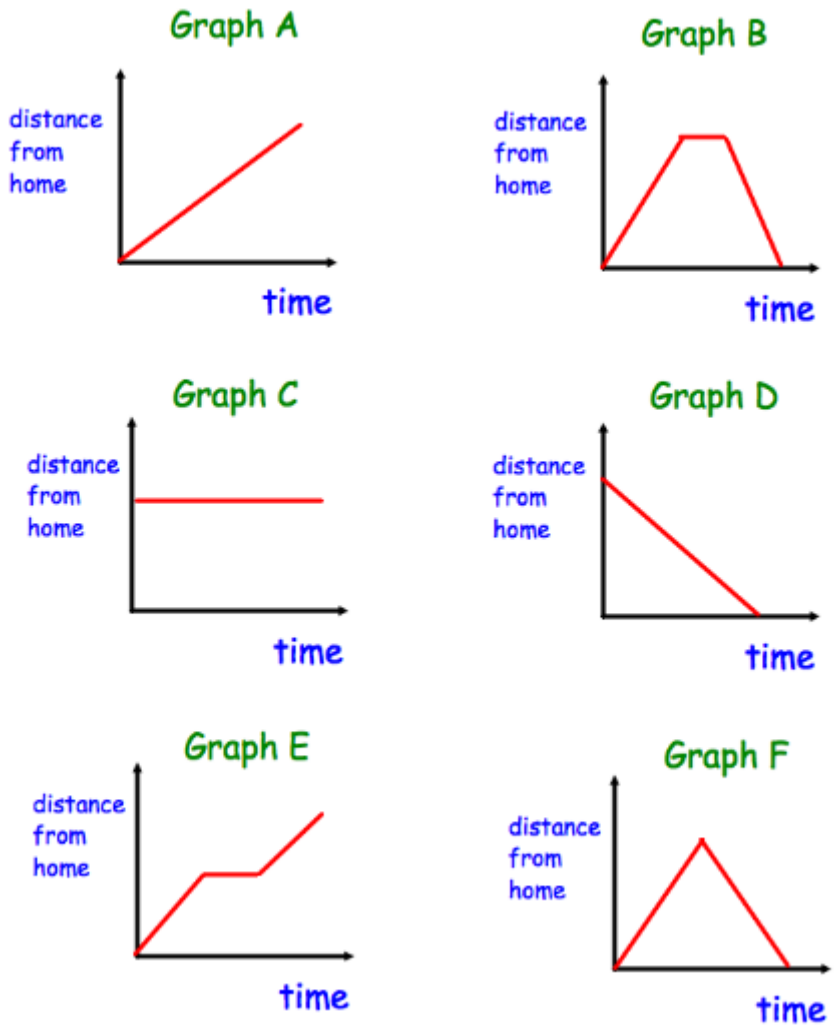
7.5miles 15miles 35miles 25miles

f) How far did Emma travel in total?

40miles 80miles 30miles none of the previous

**Question 2** *(question from CorbettMaths)*

Shown below are 6 distance-time graphs.




Each sentence in the table describes one of the graphs.
Write the letter of the correct graph next to each sentence.

1. Mr.Jones travels to work and immediately returns
2. Mr.Jones leaves work and travels home at a steady speed
3. Mr.Jones leaves home and travels to work at a steady speed
4. Mr.Jones stays at work
5. Mr.Jones travels to work, stays there for some time and then returns home
6. Mr.Jones leaves home and travels to work, stopping at the shop on the way

## B. Determining and interpreting the rate of change of linear graphs

Knowledge

Of all the graphs, linear graphs are the most studied. They are described a simple equation telling us the relation between the (dependent) quantity on the *y*-axis and the (independent) quantity on the x-axis:

$$y = mx + c$$

Thanks to this equation, we can find the value of *y* for every value of x given that we are provided with the values of the **gradient** *m* and the **intercept** c (remember that *c=0* when the line passes through the origin).

The gradient of a linear graph can be found from two points of coordinates $(x_1, y_1)$ and $(x_2, y_2)$ as

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $\Delta y$ is the change in *y* between two points, and $\Delta x$ is the change in x between the same two points. It is also called **rate of change** since it tells us how much *y* changes when x increases by one. To better see this, we can express *m* as a ratio:

$$\Delta y : \Delta x$$

Dividing both sides by $\Delta x$ we get

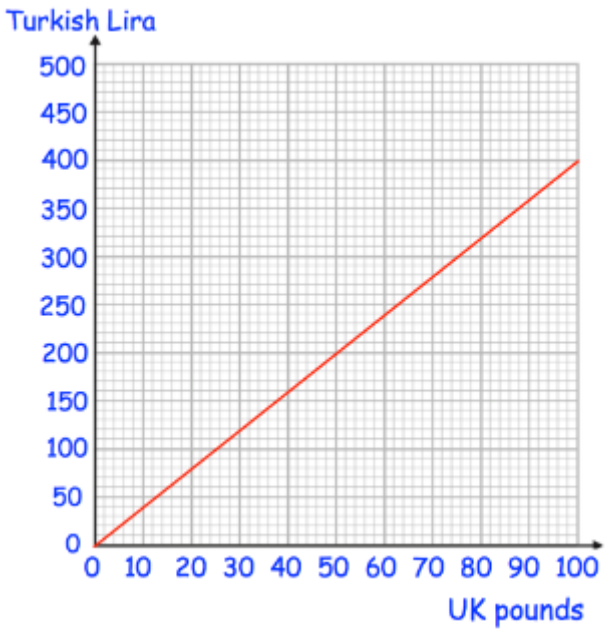
$$\frac{\Delta y}{\Delta x} : 1$$

Which tells us that when the change in x is 1, the change in *y* is $\Delta y / \Delta x$.

**Example 1** *(adapted from CorbettMaths)*

The graph below shows the conversion graph between Turkish Lira and UK pounds

a) Express the ratio between Turkish Lira and UK pounds in the form n:1.
b) Find the gradient of the line.



From the graph we can see that 400L correspond to £100. We can express this as the ratio

$$400L : £100$$

Dividing both sides by 100 we get the ratio

$$4L : £1$$

Which means that for every £1 we would get 4L.

Let's find the gradient of the line. Let's consider the two points on the line

*(0,0)* and *(100,400)*

and let's find the change in *y* and the change in *x:*

$$\Delta y = 400L - 0L = 400L$$
$$\Delta x = £100 - £0 = £100$$

The gradient of the line therefore is

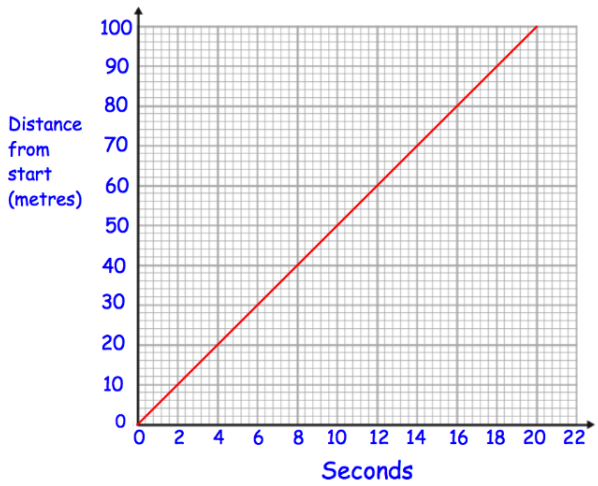
$$m = \frac{\Delta y}{\Delta x} = \frac{400L}{£100} = 4L/£$$

where we made explicit the units of the gradient.
It is important to always write the units for the rate of change!
As for the ratio above, the gradient (i.e. the rate of change) tells us that one gets 4 Turkish Lira *per* UK pound.

**Example 2** *(adapted from CorbettMaths)*

Henry participates in a 100m race. The graph below shows the distance-time graph of his race.

a) Find Henry's speed.
b) Find the gradient of the line




The speed is simply the distance covered in 1 second. We can express it as a ratio in the form *n:1*

$$100m : 20s$$

Dividing both sides by 20 we get the ratio

$$5m : 1s$$

Which means that for every second Henry travels 5m.

Let's find the gradient of the line. Let's consider the two points on the line

*(0,0)* and *(20,100)*

and let's find the change in *y* and the change in *x*:

$$\Delta y = 100m - 0m = 100m$$
$$\Delta x = 20s - 0s = 20s$$

The gradient of the line therefore is

$$m = \frac{\Delta y}{\Delta x} = \frac{100m}{20s} = 5m/s$$

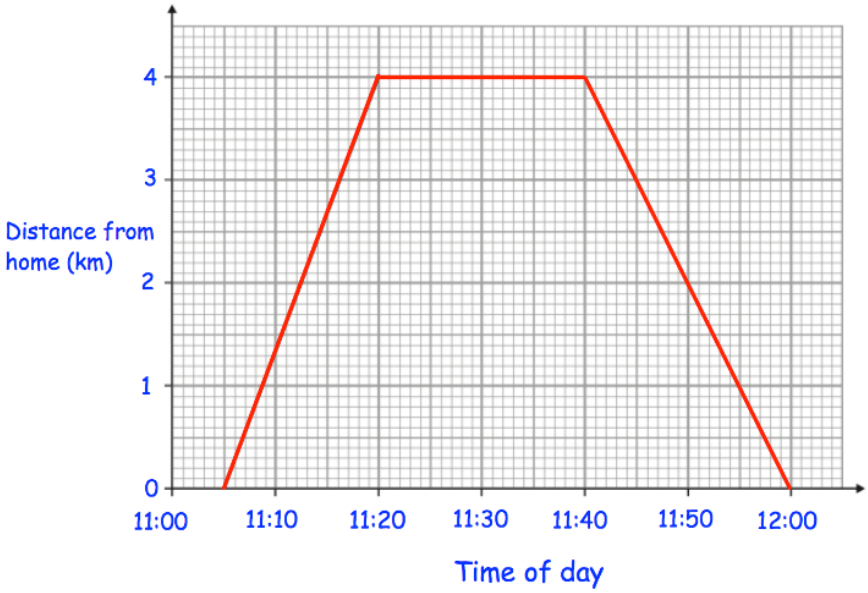
where we made explicit the units of the gradient.

It is clear from the units that the gradient represents how fast Henry is. It is important to notice that in this case the gradient might be negative: this is because the gradient represents the velocity rather than the speed!

**Example 3** *(adapted from CorbettMaths)*

Laura goes for a cycle from her house to the post office, 4km away.

a) How long did it take Laura to cycle to the post office?
b) Work out Laura's speed cycling to the post office.
c) How long did Laura spend at the post office?
d) Work out the gradient of the line representing Laura cycling back home.
e) If Laura, instead of stopping home on the way back, continued to cycle at a constant velocity, how far from home she would at 13:00?




Laura leaves home at 11:05 and arrives at the post office at 11:20. It took her 15 minutes to cycle to the post office (4km).
We can find her speed as the ratio between the distance cycled and the time:

$$4km : 15'$$

Before putting this in the form n:1, let's convert the time in hours:

$$4km : \frac{1}{4}h$$

(15 minutes is a quarter of an hour). Multiplying both sides by 4 we get the ratio

$$16km : 1h$$

Which means that for Laura would have cycled 16km if she kept cycling for 1 whole hour. Which means that her speed was

$$16km/h$$

Laura was at the post office from 11:20 to 11:40, therefore for 20 minutes.

Let's find the gradient of the line describing Laura cycling back. Let's consider the two points on the line

*(11:40,4)* and *(12:00,0)*

and let's find the change in *y* and the change in *x*:

$$\Delta y = 0km - 4km = -4km$$
$$\Delta x = 12:00 - 11:40 = 20' = \frac{1}{3}h$$

The gradient of the line therefore is

$$m = \frac{\Delta y}{\Delta x} = \frac{-4km}{\frac{1}{3}h} = \frac{-4km \times 3}{\frac{1}{3}h \times 3} = -12km/h$$

where we made explicit the units of the gradient.
This is obviously a velocity: it is negative! It is negative because Laura is cycling in the opposite direction we decided being the positive one (i.e. from home).

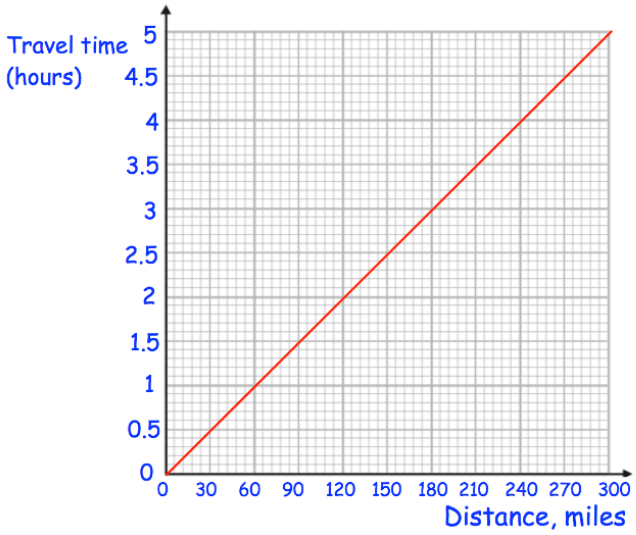
Laura is cycling at a speed of 12km/h on her way back. Therefore, if she continued cycling at the same speed rather than stopping at home, at 13:00 she would find herself 12km away from home.

Practice

**Exercise 1** *(adapted from CorbettMaths)*

The graph below represents the journey of a train.

a) Find the speed of the train.
b) Write the equation of the line.

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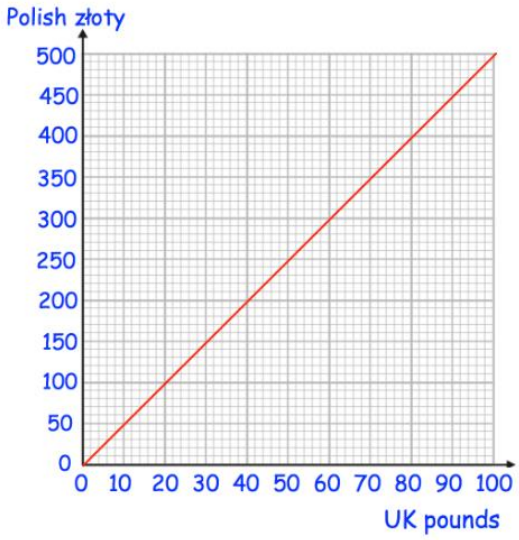
a) 60 mph b) y=60x

**Exercise 2** *(adapted from CorbettMaths)*

The graph below represents the conversion graph between Polish zloti and UK pounds

a) Find the rate of change between Polish zloti and UK pounds (expressed in the form n:1)
b) Write the equation of the line
c) Use the equation you found to convert £250 in Polish zloti



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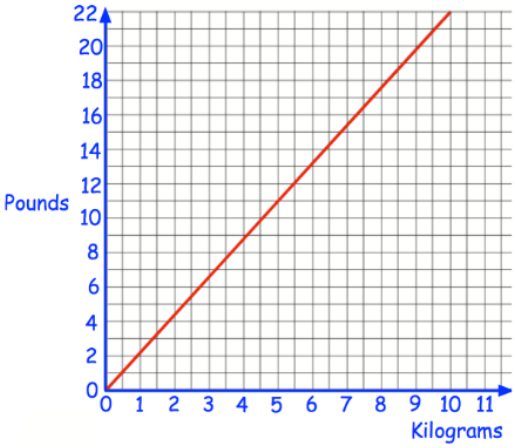
a) 5z.£1 b) y=5x c) 1250z

**Exercise 3** *(adapted from from CorbettMaths)*

The graph below represents the conversion graph between pounds and kilograms.
Jenny's weight is 65kg.

a) Find the gradient of the line (with units)
b) Write the equation of the line
c) Use the equation of the line to find Jenny's weight in pounds
d) Knowing that the conversion ratio between stones and pounds is 1 stone : 14 pounds, find Jenny's weight in stones

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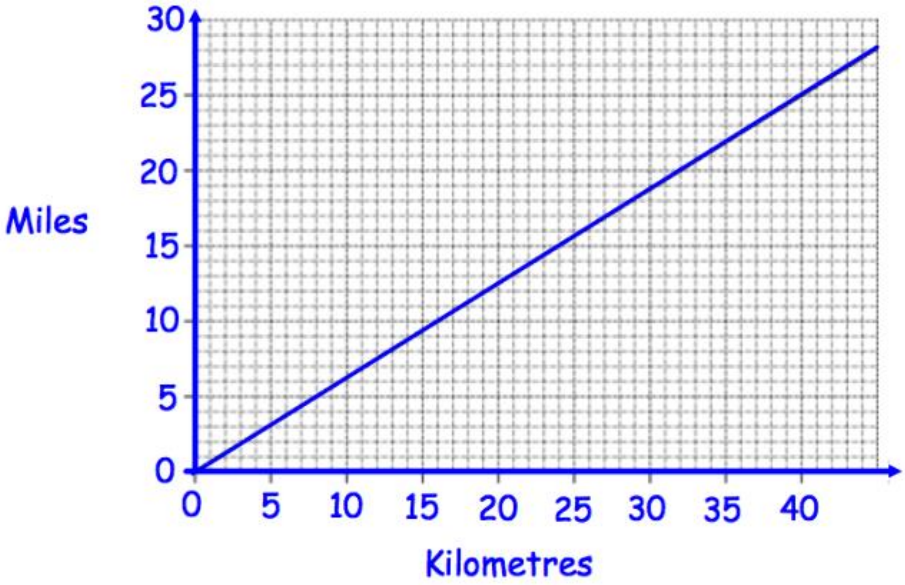
a) 2.2pounds/kg b) y=2.2x c) 143pounds d) 10.2stones (1d.p.) or 10stones 3pounds

Assessment

**Question 1** *(adapted from CorbettMaths)*

Consider the following conversion graph and answer the questions below




a) 15 miles are equivalent to

24km | 9km | 28km | none of the previous

b) 400km are equivalent to

300miles | 320miles | 250miles | none of the previous

c) The ratio of change between miles and kilometres is

40:25 | 25 miles : 40km | 40 miles : 25km | none of the previous

d) The equation of the line is

y=40x | y=1.6x | y=0.625x | none of the previous

e) The units of the gradients are

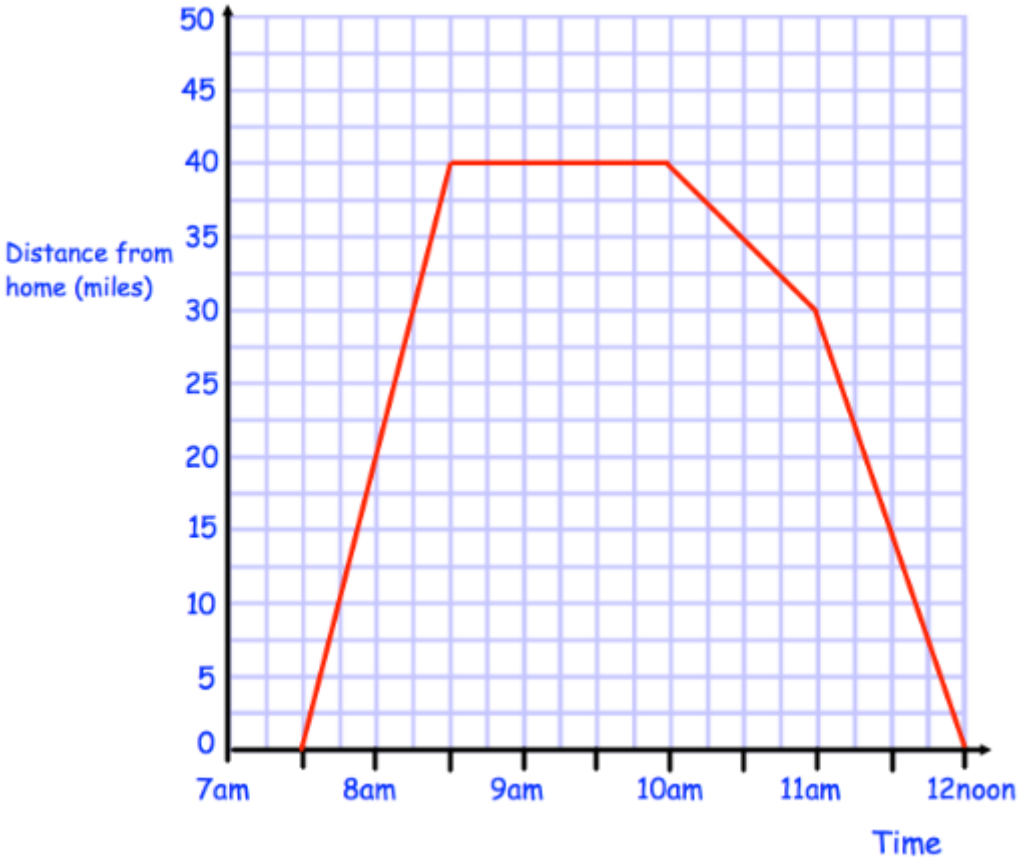
miles/km | km/miles | miles | km

f) The ratio that expresses how many kilometres correspond to one mile is

1 mile : 0.625km | 1 mile : 1.6km | 1.6 miles : 1km | 0.625 miles : 1km

**Question 2** *(adapted from CorbettMaths)*

Emma travelled to her Grandmother's house and back.
The distance-time graph shows information about her journey.




a) What is Emma's speed during the first part of her journey?

40hours/mile 26.7mph 40mph none of the previous

b) What is the equation of the line between 8:30 and 10:00?

y=0 x=40 x=0 none of the previous

c) What is the gradient of the line (with units) between 10:00 and 11:00?

10mph -10mph -0.1mph 0.1mph

d) What is Emma's speed in the final part of her journey?

30mph -30mph -40mph none of the previous

# Tutorial 2 – Finding the equation of a line

| Objectives |
|---|
| A. Finding the equation of a line from a table of values or a graph<br>B. Finding the equation of a line from worded questions |

## A. Finding the equation of a line from a table of values or a graph

**Knowledge**

In the last tutorial we have seen that the equation of a line is

$$y = mx + c$$

where *m* is the rate of change and c is the intercept. Like any equation it tells us the relationship between the quantities x and *y*. In many situations, rather than be given the equation, we are provided with a list of paired values of the two quantities or with the graph. From these we can find the equation of the line.

**Example 1**

Ava needs to call a plumber to fix a leak. She asks some friends of her to get an idea of how much she is going to spend. Three friends called John, while two friends called Ellie. Both these plumbers charge a fixed rate per hour of work on top of a fixed call-fee.

| John | |
|---|---|
| **Hours worked** | **Cost [£]** |
| 2 | 220 |
| 3.5 | 310 |
| 6 | 460 |

| Ellie | |
|---|---|
| **Hours worked** | **Cost [£]** |
| 3 | 270 |
| 1 | 170 |

a) Find an equation for the cost of hiring each of the plumbers
b) Ava believes that, to fix the leak in her house, the plumber will need to work 1 and a half hours. Who should she call?

To find the equation of a line we first notice that since we are asked to find the equation *for the cost*, the *y* quantity in this case is the cost, while the independent quantity x is the hours worked. We proceed as follows:

i) We find *m*, i.e. the rate per hour. To do this we use the formula

$$m = \frac{\Delta y}{\Delta x}$$

using any two pairs of values in the table

$$m_{John} = \frac{\Delta y_{John}}{\Delta x_{John}} = \frac{£460 - £220}{6h - 2h} = \frac{£240}{4h} = 60\ £/h$$

$$m_{Ellie} = \frac{\Delta y_{Ellie}}{\Delta x_{Ellie}} = \frac{£270 - £170}{3h - 1h} = \frac{£100}{2h} = 50\ £/h$$

ii) We find c, i.e. the fixed call-fee, we use any pair of values in the table and the rate we just found: we plug these into the general equation of a line

$$£220 = 60\ £/h \times 2h + c_{John}$$

This simple equation can be easily solved for $c_{John}$

$$£220 = £120 + c_{John}$$
$$£100 = c_{John}$$

Analogously for $c_{Ellie}$:

$$£270 = 50\ £/h \times 3h + c_{Ellie}$$
$$£220 = £150 + c_{Ellie}$$
$$£120 = c_{Ellie}$$

Therefore, the equations are

$$y_{John} = 60x + 100$$
$$y_{Ellie} = 50x + 120$$

To answer the second point of the question, we need to find how much Ava would pay with both plumbers if the work takes 1.5h as she expects:

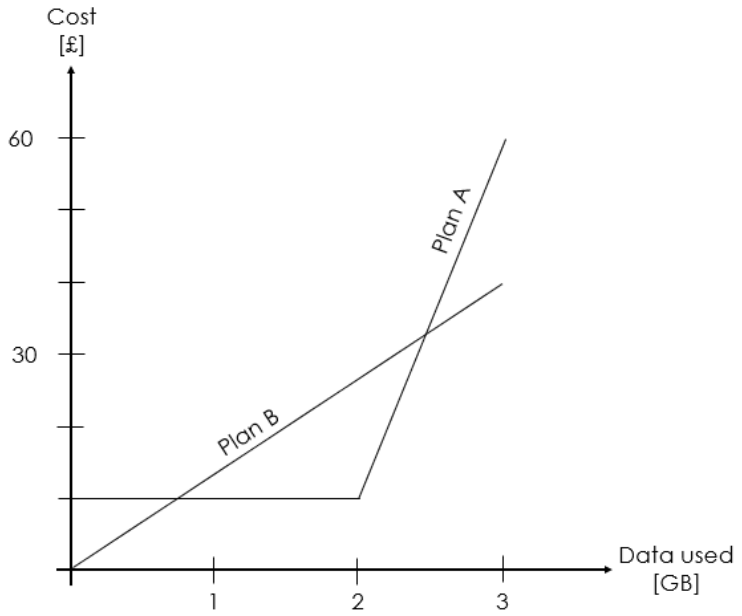
$$y_{John} = 60\ £/h \times 1.5h + £100 = £190$$
$$y_{Ellie} = 50£/h \times 1.5h + £120 = £195$$

Ava should therefore call John.

**Example 2**

Two data monthly plan costs are represented in the graph.
Plan A offers 2GB of data for £10 per month. Any extra kB of data used is paid at a fixed cost.

a) Find an equation describing the cost of Plan A after 2GB of data are used.
b) How much would one with Plan A spend if in one month uses 5GB?
c) Karen has Plan A. Last month she spent £30. How many GB of data did she use?



To find the equation of the line we proceed as we did in the previous example. We will however extract the information needed from the graph: we choose two points (2GB, £10) and (3GB, £60) to calculate the gradient of the line.

$$m_{Plan\ A} = \frac{\Delta y}{\Delta x} = \frac{£60 - £10}{3GB - 2GB} = \frac{£50}{1GB} = 50\ £/GB$$

To find c we use the equation of a general line with the gradient above and we substitute the coordinates of one of the two points above to x and y

$$£10 = 50\ £/GB \times 2GB + c$$
$$£10 = £100 + c$$
$$-£90 = c$$

The equation of the line is therefore

$$y = 50x - 90$$

We can use this to find how much one would spend when using 5GB of data:

$$y = 50\ £/GB \times 5 - £90 = £160$$

To answer the last question, we have to use the equation of the line we found above. However, in this case we are given y and we have to find x. We have to solve an equation:

$$£30 = 50\ £/GB \times x - £90$$

We add £90 to both sides of the equation

$$£120 = 50\ £/GB \times x$$

we divide by 50£/GB and we get

$$x = £120 \div 50\ £/GB = 1.4GB$$

Practice

**Exercise 1**

In the table below we have reported temperature and corresponding pressure of a perfect gas in a sealed container. The relationship between these two quantities is linear.

| Temperature [°C] | Pressure [Pa] |
|---|---|
| 0 | 10000 |
| 25 | 10915 |
| 50 | 11830 |
| 75 | 12745 |
| 100 | 13660 |

a) Find the equation for the pressure (y) in terms of the temperature (x)
b) Find the value of the pressure when the temperature is 200°C
c) Find the value of the temperature (rounded to the unit) when the pressure is 0 Pa.

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a) y=36.6x+10000 b) 28300Pa c) -273°C

**Exercise 2**

The school play is performed for 5 days in a local theatre and the net revenue will be donated to charity. The theatre is rented at a fixed price per day. Every ticket is sold at the same price. In the table below we have reported the **net** revenue (money earned – money for the rent) the show made with the number of paying spectators.

| **Paying spectators** | **Revenue [£]** |
|---|---|
| 30 | 40 |
| 47 | 91 |
| 50 | 100 |
| 51 | 103 |
| 73 | 169 |
| x | y |

a) Find the equation for the net revenue (y) in terms of the number of paying spectators (x)
b) What does the gradient represent? And the intercept?
c) What would the net revenue of the last show be if there are 75 paying spectators?
d) How many paying spectators would they need to see the last show if they want to donate a total of £600 to charity?

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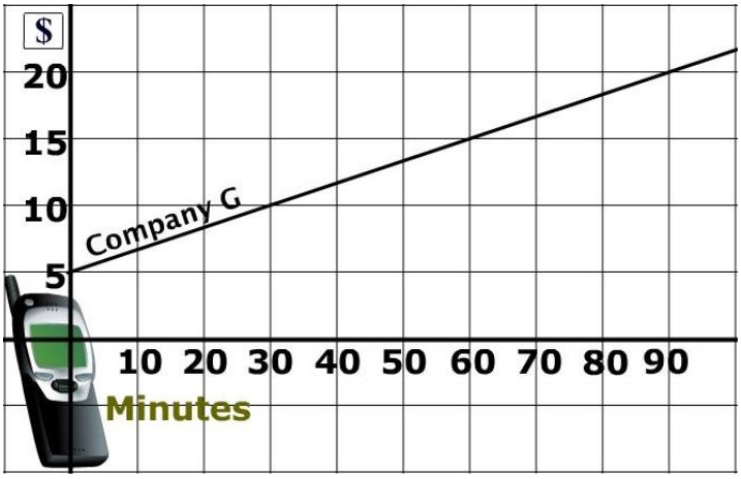
a) y=3x-50 b) The gradient represents the ticket price. The intercept represents the daily cost to rent the theatre c) £175 d) 49

**Exercise 3** *(question adapted from Mathwarehouse)*

In the graph below is represented how much Company G charges for phone calls.




a) Find the equation of the cost (y) in terms of the minutes (x) (express the gradient as a fraction in its simplest form).
b) How much would you spend if you if at the end of the month you total 42 minutes of phone calls?
c) Last month you spent $16. How many minutes of phone calls you totalled?

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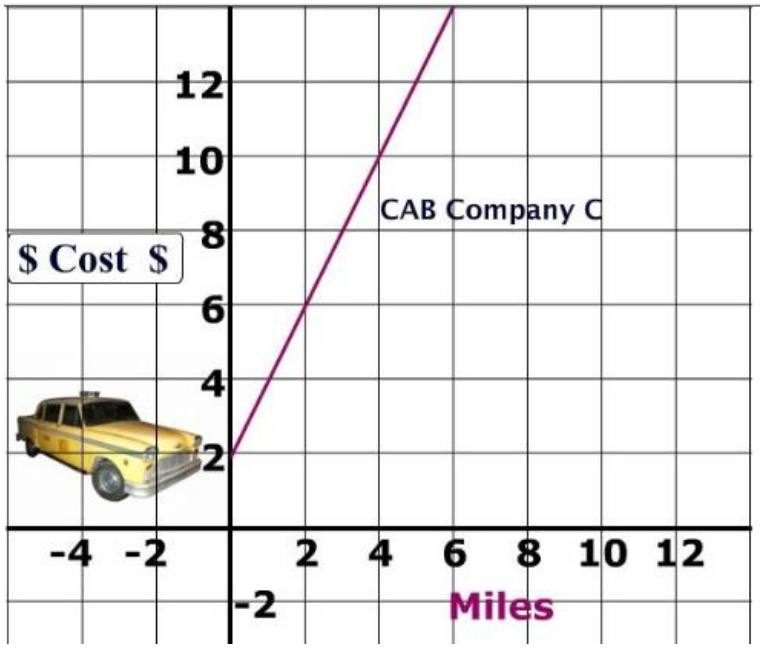
a) $y=1/6x+5$ b) $12 c) 66

**Exercise 4** *(question adapted from Mathwarehouse)*

In the graph below is represented how much a cab company charges.




a) Find the equation of the cost (y) in terms of the miles (x)
b) How much would you spend if you hire a car for a 7.3 miles journey?
c) You paid $16 for an 8 miles journey. What was the discount the driver gave you?

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a) y=2 x + 2  b) 16.6 miles  c) 2

Assessment

**Question 1**

In the table below we have reported temperature and corresponding volume of a perfect gas in piston which maintains a constant pressure. The relationship between these two quantities is linear.

| Temperature [°C] | Volume [L] |
|---|---|
| 0 | 22.386 |
| 25 | 24.436 |
| 50 | 26.486 |

a) The gradient of the line describing the volume (y) in terms of the temperature (x) is

12.195 L/°C  0.082 L/°C  12.195 °C/L  0.082 °C/L

a) The intercept of the line describing the volume (y) in terms of the temperature (x) is

22.386 L  0°C  26.486 L  none of the previous

b) The volume of the gas when the temperature is -25°C is

20.336L  2.05L  24.436L  none of the previous

c) The temperature of the gas, when the volume is 28.536L, is

80°C  621°C  75°C  none of the previous

**Question 2**

In the graph below, two monthly plans for Company S and Company T are represented.

a) The gradient of the line describing the cost (y) in terms of the minutes (x) for Company S is

0.25 $/min  4 $/min  0.75 $/min  1.333 $/min

a) The gradient of the line describing the cost (y) in terms of the minutes (x) for Company T is

0.25 $/min  4 $/min  0.75 $/min  1.333 $/min

b) The fixed cost one has to pay to activate Company T's plan is

$15  $20  $5  none of the previous

c) With Company S, for 60 minutes one would spend

$25  $30  $35  none of the previous

## B. Finding the equation of a line from worded questions

Knowledge

When facing a worded question, we have to separate the data that we are given from the quantities that we need to find. In the cases we are going to see in this unit we will have to write the equation of a line. It is therefore crucial that we identify what the x and *y* quantities are, what the gradient is, and what the intercept is.

x *and* y *quantities:* the *y* quantity is **"written in terms of" / "based on"** the x quantity. In many cases we are asked to find a certain quantity when we are not given a value for the other, but rather an unknown: the quantity to find is *y*, the unknown quantity is x.

*Gradient*: this is usually expressed as a rate. Look out for the words "per" and "every": e.g. pounds **per** minute, miles **per** hour, dollars **per** metre, cost **per** person etc, **every** day, **every** minutes, **every** candy sold etc.

*Intercept*: this is the value of the *y* quantity when the x quantity is zero. It could be for example a **fixed charge / activation or initial cost** (something you pay without "consuming" anything).

**Example 1** *(question from Mathwarehouse)*

Lin is tracking the progress of her plant's growth. Today the plant is 5 cm high. The plant grows 1.5 cm per day.

a) Write a linear model that represents the height of the plant after *d* days.
b) What will the height of the plant be after 20 days?

From question *a* we understand that the *y* quantity will be the height of the plant and that the x quantity is the number of days (this is given as an unknown *d*; we will write this letter in place of x).

From the text it is clear what the gradient is: "1.5 cm **per** day"

$$m = 1.5cm/day$$

The intercept is 5cm, i.e. the height of the plant after *d=0* days (in other words: today).

$$c = 5cm$$

The linear model (the equation of the line) that represent the height of the plant after *d* days is therefore:

$$y = 1.5\,d + 5$$

Point *b* is now easily answered by using the equation of the line:

$$height\ after\ 5\ days = 1.5\ cm/day \times 20days\ + 5cm = 35cm$$

**Example 2** *(question from Mathwarehouse)*

Conner has $25000 in his bank account. Every month he spends $1500. He does not add money to the account.

a) Write a linear model that shows how much money will be in the account after x months.
b) How much money will Conner have in his account after 8 months?

From question *a* we understand that the *y* quantity will be the money in the account and that the x quantity is the number of month (this is given as an unknown x).

From the text we know that Connor spends $1500 **every** month. We have to be careful here: since this amount is spent, it will decrease the amount in the bank account, it is therefore a negative rate

$$m = -1500\$/month$$

The intercept is $25000cm, i.e. the money he has after x=*0* months (in other words: the initial amount).

$$c = \$25000$$

The linear model (the equation of the line) that represent how much money will be in the account after x months is therefore:

$$y = -1500x + 25000$$

Point *b* is now easily answered by using the equation of the line:

$$money\ after\ 8\ months = -1500\$/month \times 8months\ + \$25000 = \$13000$$

Practice

**Exercise 1** *(question from Mathwarehouse)*

Paul opens a savings account with $350. He saves $150 per month. Assume that he does not withdraw money or make any additional deposits.

a) Write a linear model that represents the total amount of money Paul deposits into his account after *m* months.
b) After how many months will Paul have more than $2000?

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b) y=150*m*+350 b) 11 months

**Exercise 2** *(question from Mathwarehouse)*

A salesperson receives a base salary of $35000 and a commission of 10% of the total sales for the year.

a) Write a linear model that shows the salesperson's total income based on total sales of *k* dollars.
b) If the salesperson sells $250000 worth of merchandise, what is her total income for the year, including her base salary?

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a) $y=0.1k+35000$ b) $60000

**Exercise 3** *(question from Mathwarehouse)*

Mr. Thompson is on a diet. He currently weighs 260 pounds. He loses 4 pounds per month.

a) Write a linear model that represents Mr. Thompson's weight after *m* months.
b) After how many months will Mr. Thompson reach his goal weight of 220 pounds?

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a) $y=-4m+260$ b) 10 months

**Exercise 4** *(question from Mathwarehouse)*

The population of Bay Village is 35000 today. Every year the population of Bay Village increases by 750 people.

a) Write a linear model that represents the population of Bay Village x years from today.
b) In approximately how many years will the population of Bay Village exceed 50,000 people?

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a) $y=750x+35000$ b) 20 years

**Challenge: Exercise 5** *(question from Mathwarehouse)*

Amery has x books that weigh 2 pounds each and y books that weigh 3 books each. The total weight of his books is 60 pounds.

a) Write a linear model that relates the number of 2 pound books to the number of 3 pound books Amery has.
b) If Amery has 10 3-pound books, how many 2-pound books does he have?

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a) $2x+3y=60$ b) 15

Assessment

**Question 1**

Kara used the linear model y= 20000 + 0.3x to predict her total salary from achieving total sales of x.

a) Her base salary is

| | | | |
|---|---|---|---|
| 20003.6 | 0.3 | 20000 | 20000.3 |

b) What percent commission does she earn?

| | | | |
|---|---|---|---|
| 0.3 | 3% | 30% | 200% |

c) If she achieves total sales of $10000, what is her salary at the end of the year?

| | | | |
|---|---|---|---|
| 30000 | 23000 | 20300 | none of the previous |

d) If at the end of the year she gets a total salary of $29000, what is the total sales she achieved?

| | | | |
|---|---|---|---|
| 28700 | 14700 | 35000 | none of the previous |

**Question 2**

A cab company charges a $4 boarding rate in addition to its meter which is $ ¾ for every mile.

a) The gradient of the line describing the cost in terms of the miles is

| | | | |
|---|---|---|---|
| 0.75 $/mile | 4 $/mile | 3.25 $/mile | none of the previous |

a) The intercept of the line describing the cost in terms of the miles is

| | | | |
|---|---|---|---|
| $¾ | $4 | $4.75 | none of the previous |

b) You travel 20 miles. How much do you spend?

| | | | |
|---|---|---|---|
| $18 | $19 | $4.752 | none of the previous |

a) You pay $16 for a ride. How many miles have you travelled?

| | | | |
|---|---|---|---|
| 16 miles | 9 miles | 12 miles | none of the previous |

# Tutorial 3 – Average & instantaneous rate of change

| Objectives |
|---|
| A. Finding the average rate of change from a graph<br>B. Finding the instantaneous rate of change from a graph |

## A. Finding the average rate of change from a graph

| Knowledge |
|---|

Imagine to be travelling by car. The journey lasts 1 hour and the distance travelled is 40 miles. Can we say that the speed at which you travelled is 40 miles per hour? It seems reasonable. However, during your journey you stopped at some traffic lights (which means that you were not moving and therefore your speed was zero) and you definitely remember that on the motorway you have seen the speedometer reaching 70.

So, does it really make sense to say that the speed was 40mph? Yes, it does! 40mph was your **average speed**! This quantity tells us that **if you were travelling at a constant speed**, to travel the same distance in the same time, this speed would have been 40 miles per hour.

In general, we can find the **average rate of change between two points** on a graph as the gradient of the line passing through those two points.

**Example 1** *(question adapted from Corbettmaths)*

Jack is filling a container with water.
The graph shows the depth of the water, in centimetres, t seconds after the start of filling the container.

a) What is the rate at which, on average, the container is filled during the first 25 seconds?

To answer the question, we need to find the gradient of the line passing through the points of the graph at t=0 and t=25. These two points have coordinates (0,0) and (25,4.25). The gradient is therefore:

$$m = \frac{\Delta y}{\Delta x} = \frac{4.25cm - 0cm}{25s - 0s} = 0.17\ cm/s$$

This tells us that **on average** the depth of the water in the container was rising 0.17cm every second.
This is equivalent to say that if Jack was filling the container at the constant rate of 0.17cm/s, after 25 seconds, the depth of the water would be the same as the depth in the scenario represented in the graph.

**Example 2** *(question adapted from Corbettmaths)*

A rocket is fired upwards as part of a science experiment.
The graph shows the height of the rocket above the ground.

a) Find the average speed of the rocket between 0 and 2.5 seconds.
b) Find the average speed of the rocket between 2.5 and 4.5 seconds.
c) Find the average speed of the rocket between 0 and 4.5 seconds.

To answer question a, we need to find the gradient of the line passing through the points of the graph at t=0 and t=2.5. These two points have coordinates (0,0) and (2.5,24). The gradient is therefore:

$$m = \frac{\Delta y}{\Delta x} = \frac{24m - 0m}{2.5s - 0s} = 9.6\ m/s$$

This tells us that **on average** the rocket was travelling up 9.6m every second.

Similarly, to answer question b, we need to find the gradient of the line passing through the points of the graph at t=2.5 and t=4.5. These two points have coordinates (2.5,24) and (4.5,30.4). The gradient is therefore:

$$m = \frac{\Delta y}{\Delta x} = \frac{30.4m - 24m}{4.5s - 2.5s} = 3.2\ m/s$$

This tells us that **on average**, between 2.5 and 4.5 seconds, the rocket was travelling up 3.2m every second.

In this second time interval, the rocket was moving three times slower than in the previous time interval.

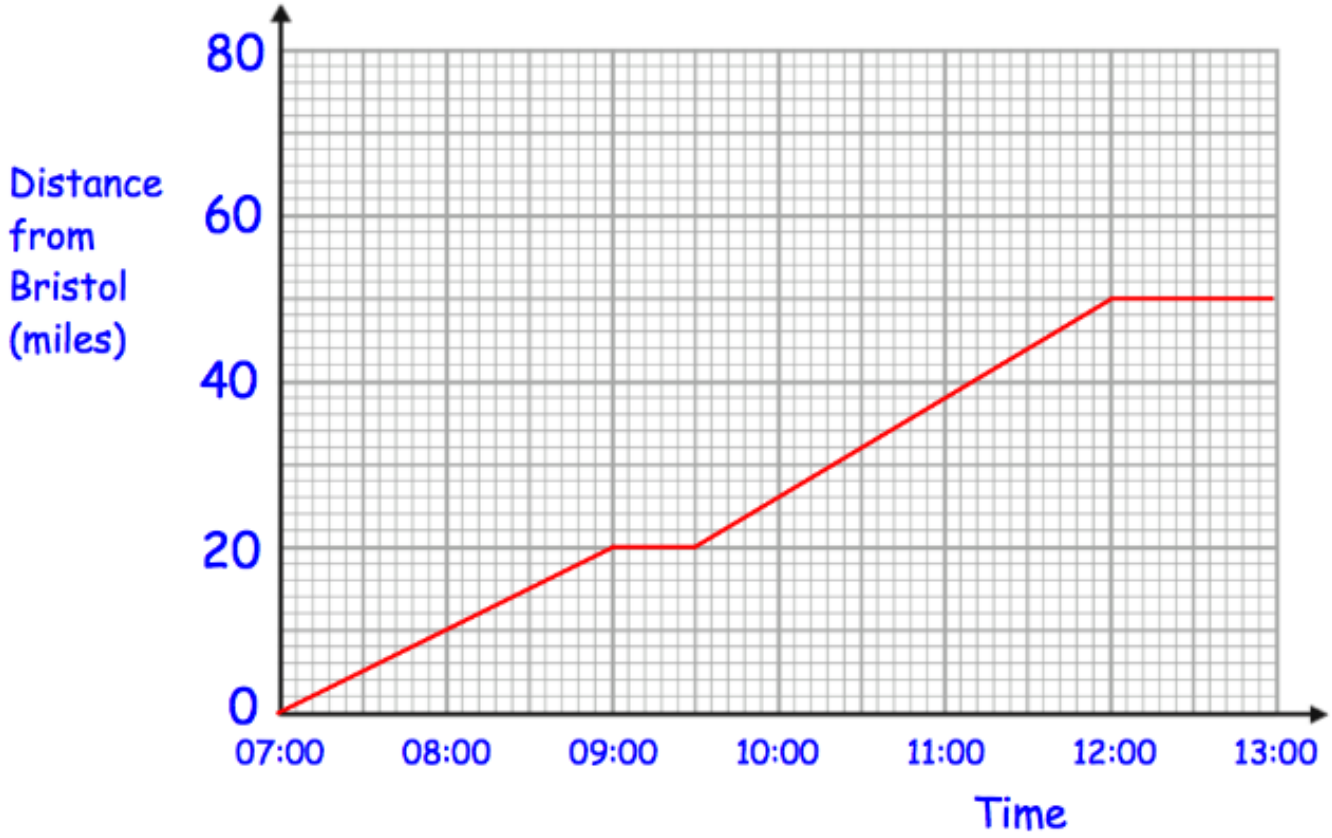
What do you expect the average speed to be between 0 and 4.5 seconds?

On average it is slower than in the first 2.5 seconds, but faster than between 2.5 and 4.5 seconds: $m = \frac{30.4m - 0m}{4.5s - 0s} = 6.76\ m/s$

Practice

**Exercise 1** *(question adapted from Corbettmaths)*

Anne cycles from Bristol to Salisbury.
The diagram shows the distance-time graph of her journey.




a) What was Anne's average speed (from when she started to the moment she arrived)?
b) At what time would she have arrived if she cycled at her initial speed without stopping?

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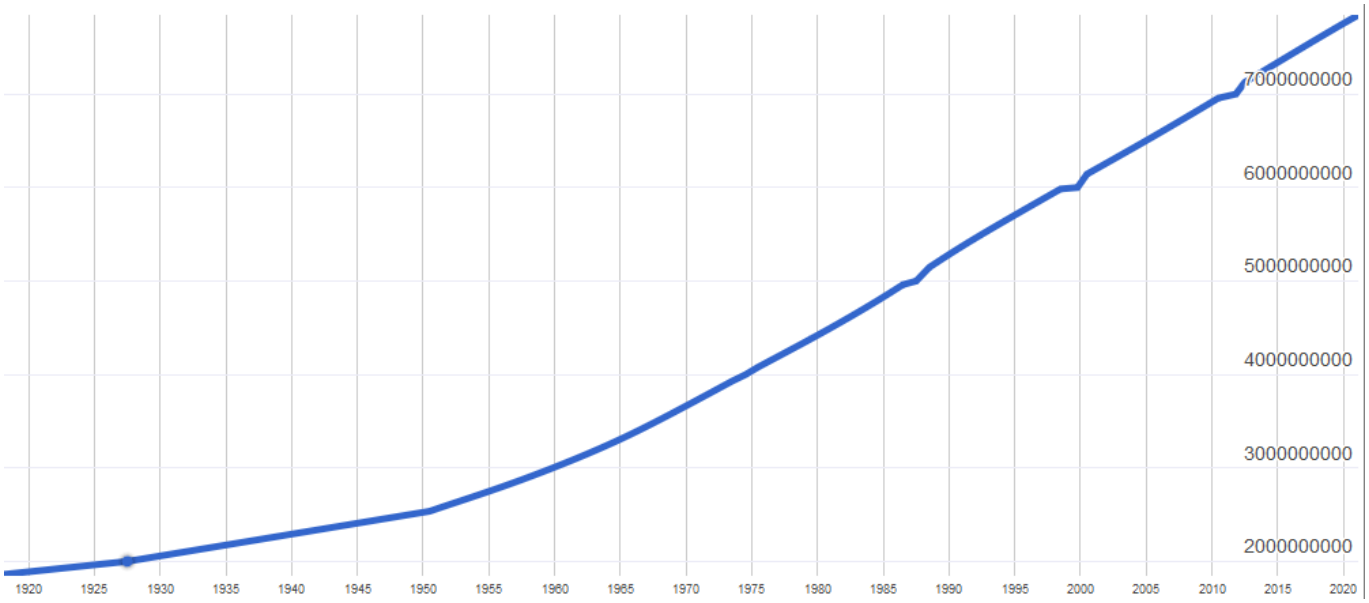
a) 10 mph b) 12:00

**Exercise 2**

The graph shows the world population in the last 100 years.




a) At what rate has the world population been growing, on average, in the last ten years?
b) At what rate did the world population grow, on average, between 1920 and 1940?

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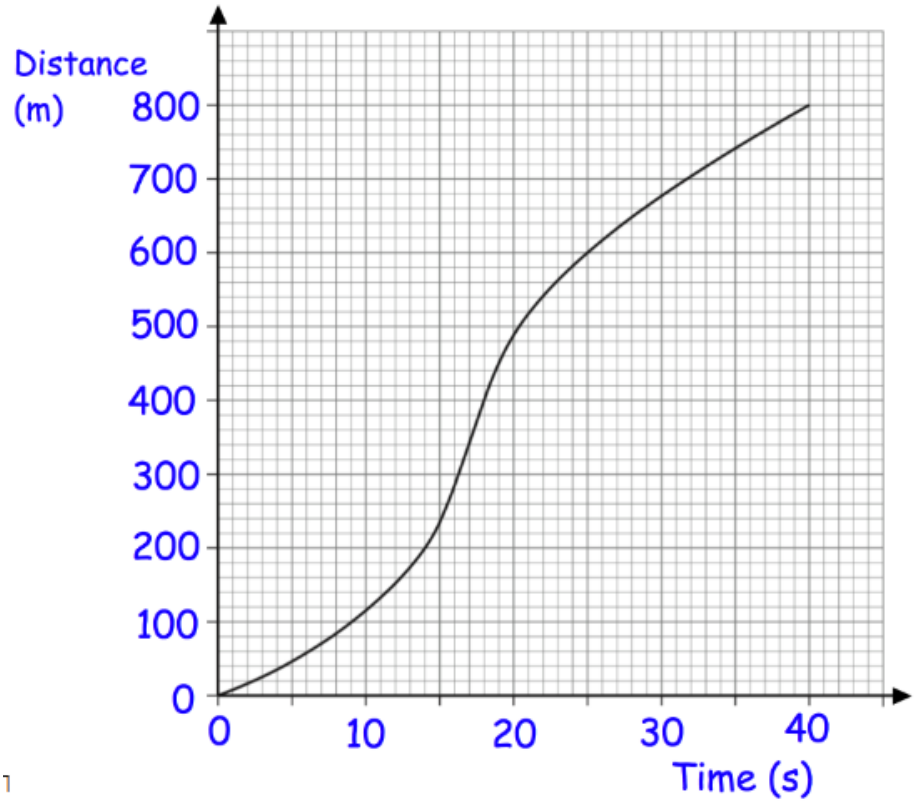
a) 100,000,000 per year b) 20,000,000 per year

**Exercise 3** *(question adapted from Corbettmaths)*

Below is the distance-time graph for the first 40 seconds of a train journey.




c) Find the average speed of the train between 0 and 10 seconds.
d) Find the average speed of the train between 10 and 20 seconds.
e) Find the average speed of the train between 20 and 30 seconds.
f) Find the average speed of the train between 30 and 40 seconds.

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a) 12 m/s b) 36 m/s c) 20 m/s d) 12 m/s

Assessment

**Question 1** *(question adapted from CorbettMaths)*

Shown is a distance-time graph of part of a journey.

a) The average speed of the part of journey shown in the graph is

0.77m/s 1.3m/s 1.5m/s none of the previous

b) In what time interval is the average speed higher?

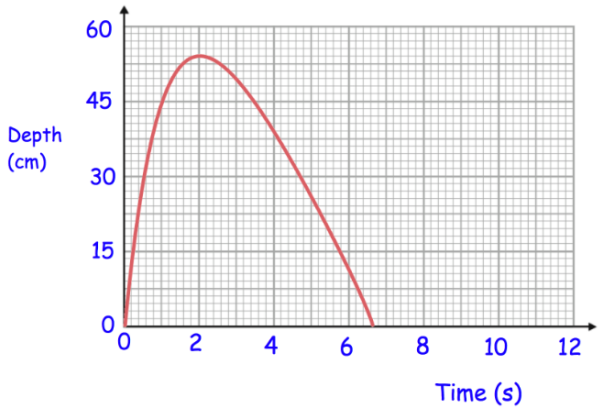
Between 0 and 5 seconds Between 0 and 10 seconds Between 10 and 15 seconds Between 5 and 30 seconds

c) In what time interval is the average speed lower?

Between 0 and 5 seconds Between 5 and 10 seconds Between 10 and 15 seconds Between 5 and 30 seconds

**Question 2** *(question adapted from CorbettMaths)*

Hugh has a bucket with holes in it. Hugh fills the bucket with water and records the depth of water.
The graph shows the depth of water in the bucket.




a) The average rate of change of depth of water between 0 and 2 seconds is

27cm/s 0.037cm/s 54cm/s none of the previous

b) The average rate of change of depth of water between 2 and 6 seconds is

-10.875cm/s 0.092cm/s -0.092cm/s none of the previous

a) The average rate of change of depth of water between 1 and 3.6 seconds is

0cm/s 15cm/s 11.25cm/s -11.25cm/s

## B. Finding the instantaneous rate of change from a graph

Knowledge

Let's consider the same car journey we thought of in the previous section: the journey lasts 1 hour and the distance travelled is 40 miles. The average speed was 40mph but as we already noticed during your journey you stopped at some traffic lights (which means that you were not moving and therefore your speed was zero) and you definitely remember that on the motorway you have seen the speedometer reaching 70. The speed seen on the speedometer "in real time" is the **instantaneous speed.** If the speedometer reads 70mph in a particular instant, that means that if you were continue travelling from that instant onward at that same speed, after one hour you would have travelled 70 miles.

We have seen in the previous section that in order to find the average speed (rate of change) between two points we have to calculate the gradient of the line passing through those two points. But what does it mean to find the rate of change in a given instant, i.e. in one point? Well, that corresponds to find the gradient of the line **tangent** (it touches the curve in only one point) to the graph in that given point!

Therefore, when we are asked to find the instantaneous rate of change at a given point we have to:

1. Draw the line tangent to the graph in that point
2. Find the gradient of that tangent line.

**Remember:**

***a)*** ***velocity*** is the rate of change of distance over time,
***b)*** ***acceleration*** is the rate of change of velocity over time,
***c)*** ***flow*** is the rate of change of volume over time.

**Example 1** *(adapted from CorbettMaths)*

The graph below shows the height of a ball above the ground after it has been launched upwards.

a) Use the graph to estimate the speed of the ball after 1 second
b) When was the speed 0m/s?

To find what the speed of the ball was after 1 second, we draw the tangent to the graph in t=1 (see black dotted line in the graph above). This line passes through the points (1,9) and (3,25.2). The gradient is therefore:

$$m = \frac{\Delta y}{\Delta x} = \frac{25.2m - 9m}{3s - 1s} = 8.1\ m/s$$

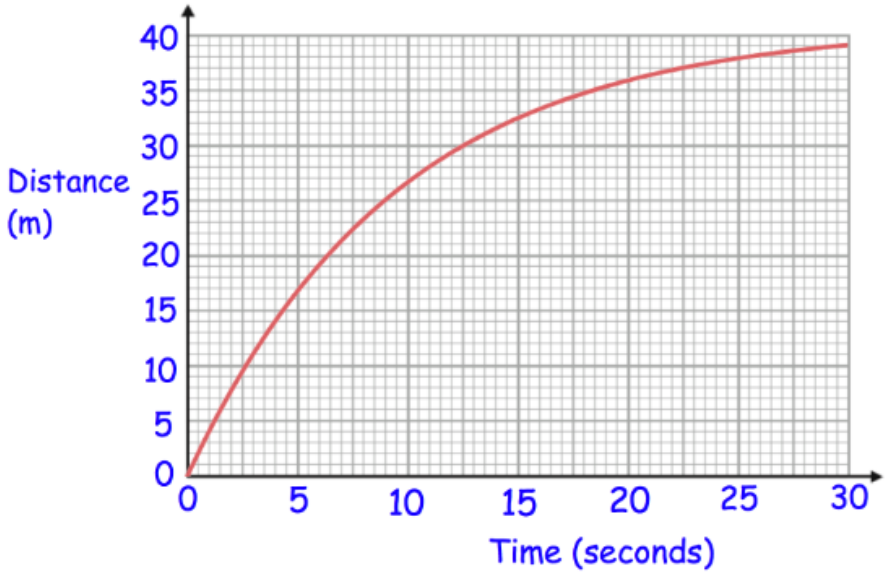
Which means that if after one second the ball continued travelling (upwards) at a constant speed (rather than being slowed down by gravity), at 3 seconds it would be 25.2m above the ground.

To answer part b, we have to remember that a line with a 0 gradient (rate of change) is horizontal, i.e. parallel to the x-axis. We see that the only point where the line tangent to the graph is horizontal is at **t=4.5s** (see blue dotted line).

Practice

**Exercise 1** *(adapted from CorbettMaths)*

Shown is a distance-time graph of part of a journey




a) Work out the initial (t=0) instantaneous velocity
b) Work out the velocity after 10 seconds.

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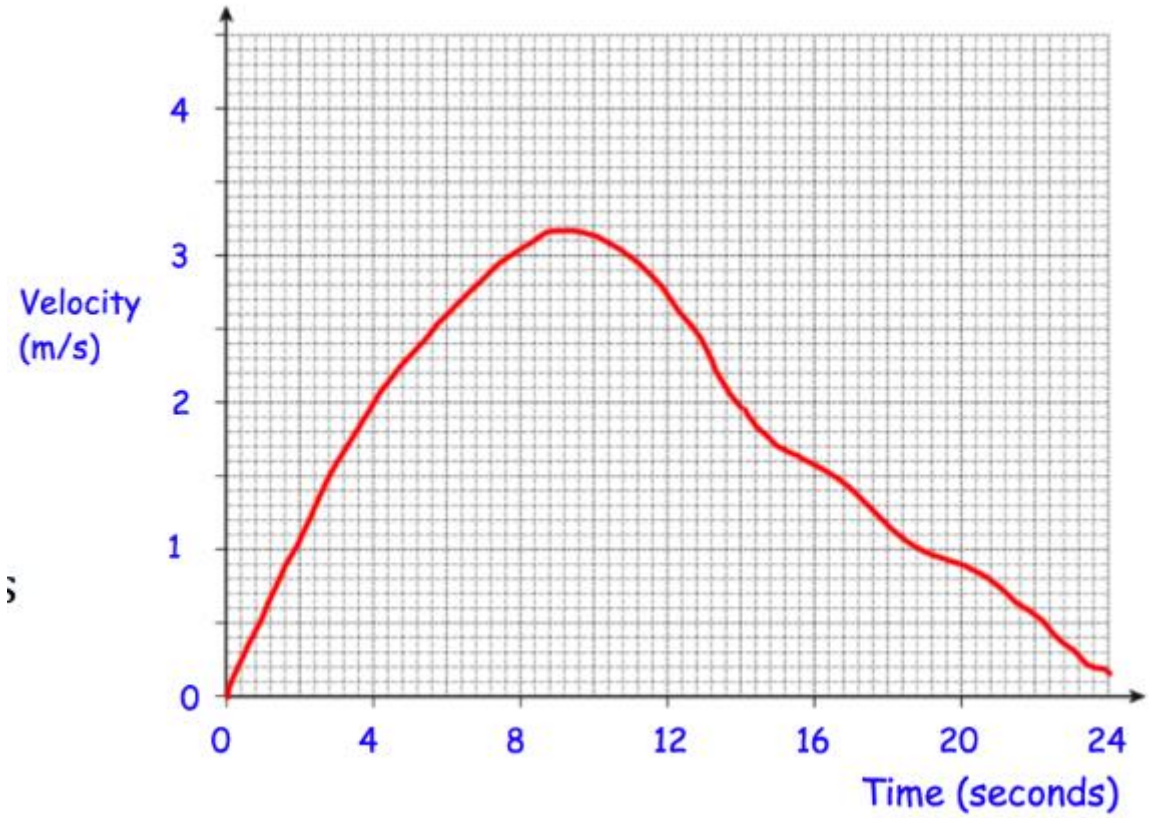
a) 5m/s b) 1.4m/s

**Exercise 2** *(adapted from CorbettMaths)*

The graph shows the velocity of an object over time




a) Use the graph to estimate the acceleration of the object after 2 seconds
b) When is the acceleration 0 $m/s^2$?
c) Use the graph to estimate the deceleration of the object at t=16s

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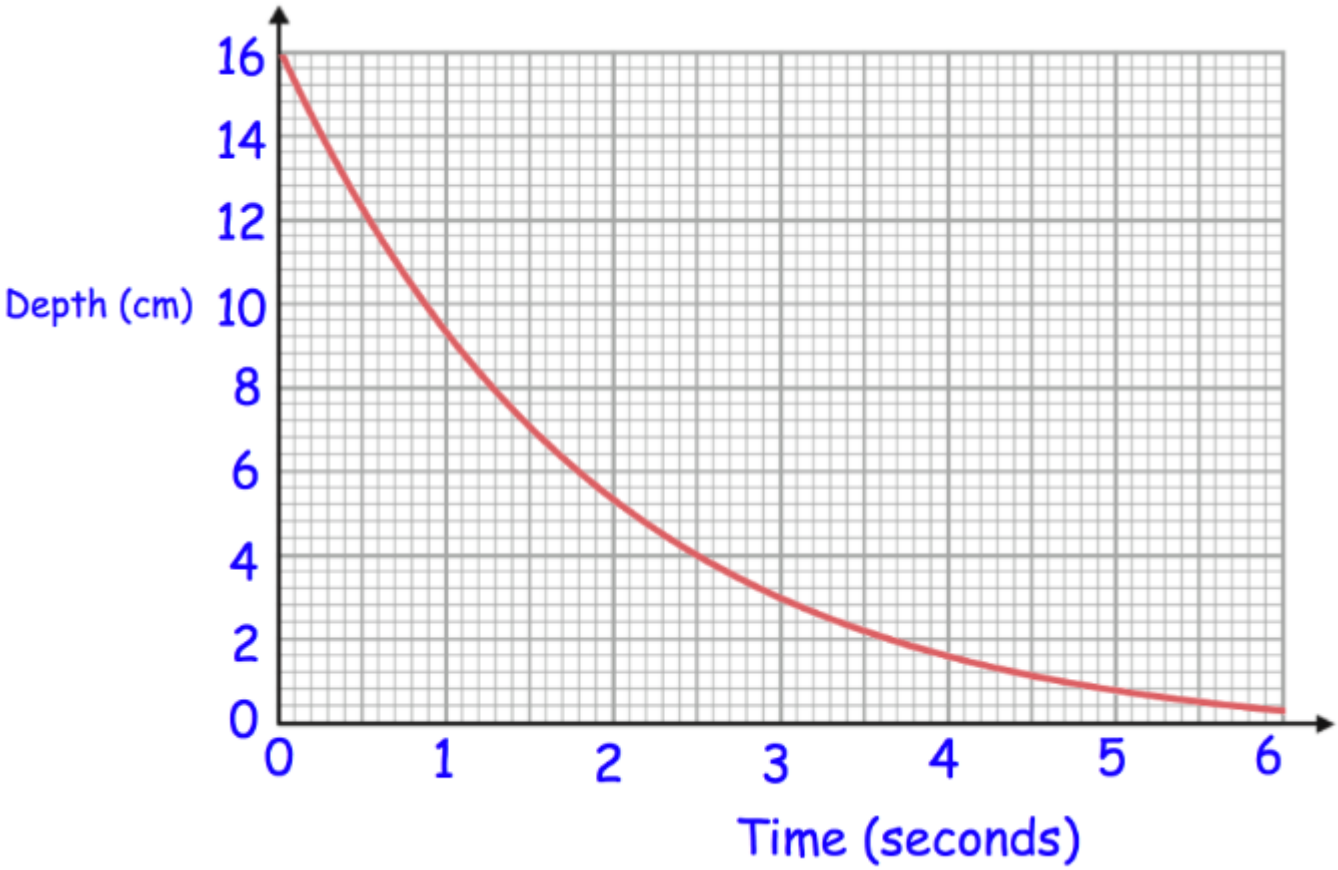
a) $0.55m/s^2$ b) 9.2s c) $-0.125m/s^2$

**Exercise 3** *(adapted from CorbettMaths)*

Water is being emptied out of a bath.
The graph shows the depth of the water for the first 6 seconds.




a) Use the graph to work out an estimate of the rate of decrease of depth at 1 second.
b) Use the graph to work out an estimate of the rate of decrease of depth at 2 second.
c) Use the graph to work out an estimate of the rate of decrease of depth at 3 second.

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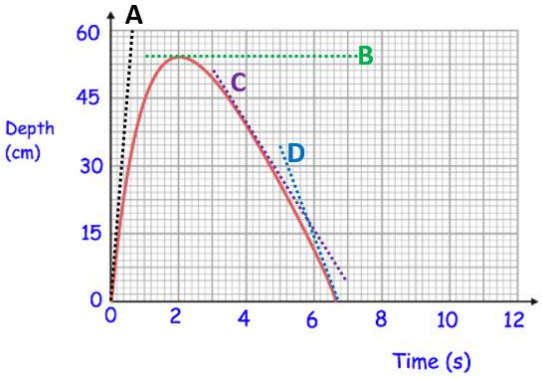
a) -5.2cm/s b) -2.8cm/s c) -2cm/s

Assessment

**Question 1** *(question adapted from CorbettMaths)*




a) To estimate the instantaneous rate of change of the depth of water at t=4s one should use line

A B C D

b) One finds that the instantaneous rate of the depth of water is 0cm/s by using line

A B C D

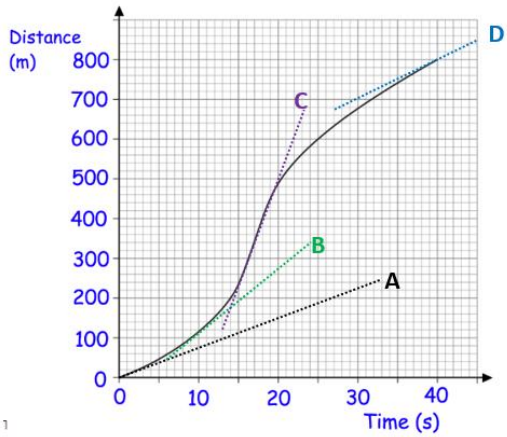
c) The instantaneous rate of change of the depth of water is 0cm/s at time

0s 2s 4s none of the previous

d) The instantaneous rate of change at t=6.6s is

positive zero negative impossible to tell

**Question 2** *(question adapted from CorbettMaths)*




a) The instantaneous velocity is 0m/s

at t=0s at t=10s at t=17.5s never

b) The line that allows us to find when the velocity reaches its highest value is

A B C D

a) The line that allows us to find when the velocity is at its lowest value is

A B C D

b) The velocity between 0 and 40 seconds is

always negative always positive initially positive, then negative initially negative, then positive

# Tutorial 4 – Filling up containers

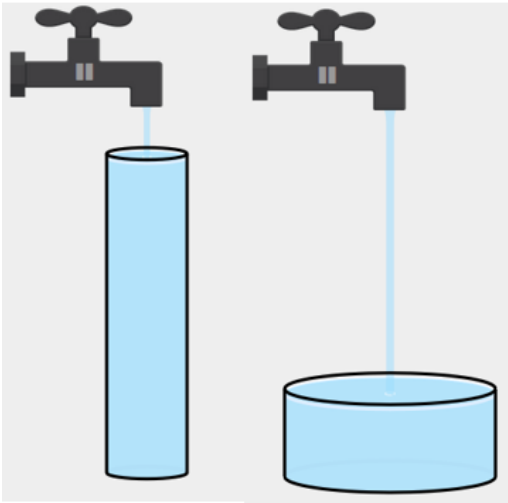
| Objectives |
| --- |
| A. Understanding why the water height in a container filled up with a constant flow of water rises quicker or slower depending on the cross-section of the container<br>B. Identifying the container from the graph representing the height of water over time |

## A. Height of water in a container

| Knowledge |
| --- |

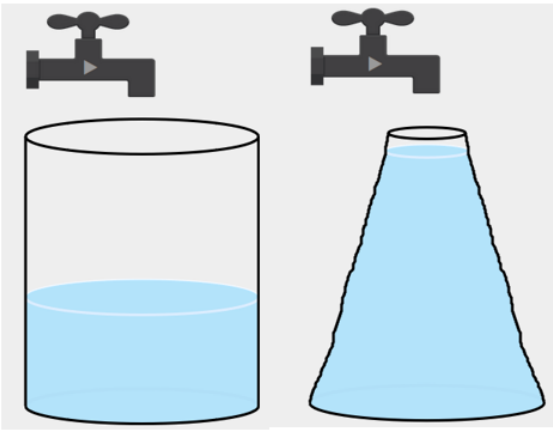
Let's consider the following situation: we have two containers of water and we are filling them with water. To do this we place them under a tap which release water at a constant flow: $1000cm^3$/s (1L/s). This means that after 1 second, in the container we have $1000cm^3$ of water.

Below we can see the two containers after 1 second:

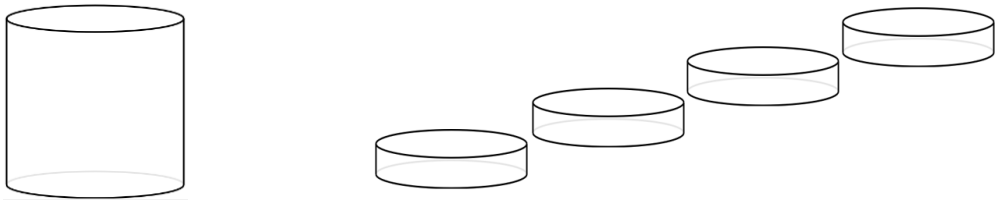


Because the volume of water poured in both container after 1s is the same ($1000cm^3$) and since the container on the left is narrower (it has a smaller cross section) than the container on the right, the height of water in the former is higher than the height of water in the latter.

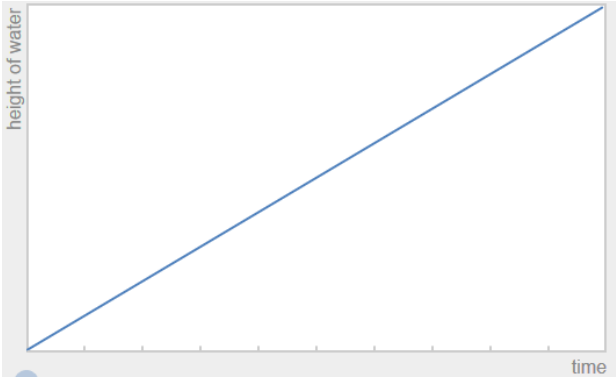
With a similar reasoning, given two containers, we can find out which one of the two will fill faster. Consider the two container below: the first one has a bigger volume than the second, therefore it will take longer to fill it



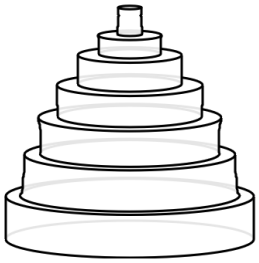
It is also important to notice that when the cross-section of a container is constant (it is a cylinder or a parallelepiped), then the height of water in the container increases at a constant rate. This is because we can "slice" our container in **equal height slices**, which means that they all have the **same volume**, and will therefore fill in the same time: for example, 1 slice will fill in one second, so 2 slices will fill in to seconds, etc.



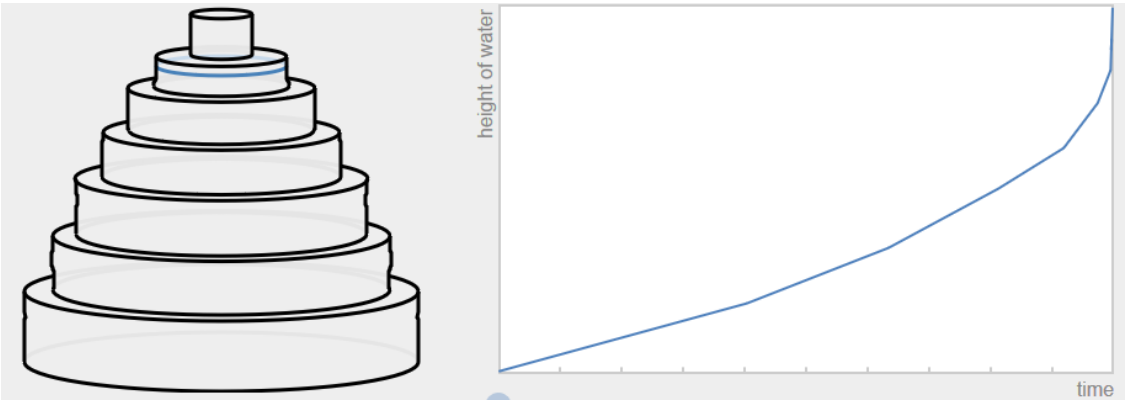
Which means that if we plot the graph of the height of water in the container over a period of time, we have a straight line




On the other hand, if the cross-section changes, then the rate changes too. Let's consider the nuptial cake-style container below



The first slice, having a bigger volume will fill up slower than the second slice. The second slice will fill up slower than the third because it has a bigger volume. Etc.
Since each slice is a cylinder, it will fill up at a constant rate. The height of the water in the container above over time is therefore represented by a series of lines each steeper than the previous:




When we have a container with an irregular cross section, we can imagine slicing it into many very short cylinders of different cross section and stack them one above the other: each cylinder will fill up at a different rate, the larger the cylinder, the slowest it will fill up, the less steep the graph will be in that point, on the other hand, a very narrow cylinder will fill up very quickly and the steepest the graph will be.

| Practice & Assessment |
| --- |
| Go to<br><br>https://student.desmos.com<br><br>and type in the class code on the board. |

## B. Identifying the container from the graph

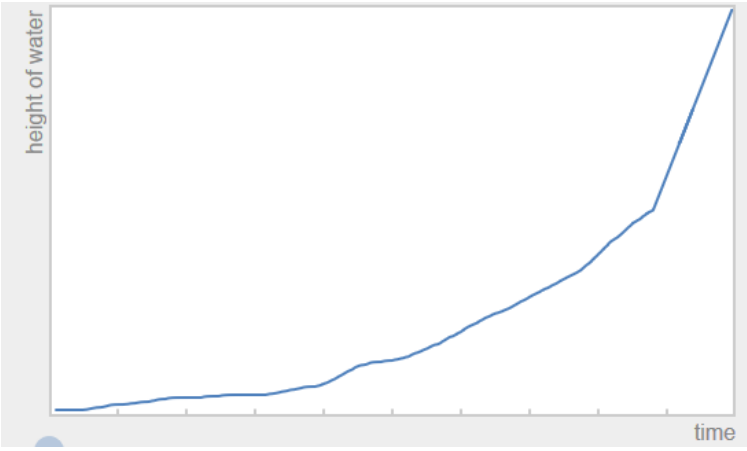
| Knowledge |
| --- |

We have seen that the narrower the cross section, the faster the container fills up. Moreover, if the cross-section changes, so does the instantaneous rate of change of the height of water: the narrower the container, the steeper the graph is.

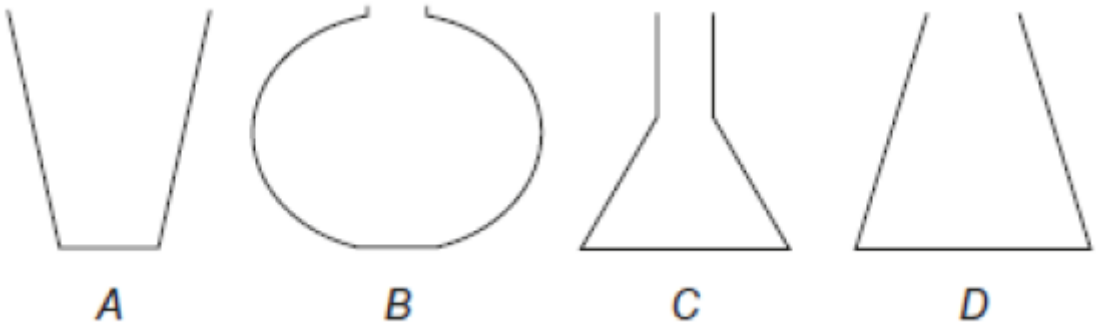
Thanks to these facts, we can identify the container when we are given the graph of how the height of the water changes over time.

**Example 1**

This is a graph of the height of water in a container against time.




Which one of the container below is the one that it is filled up with water?



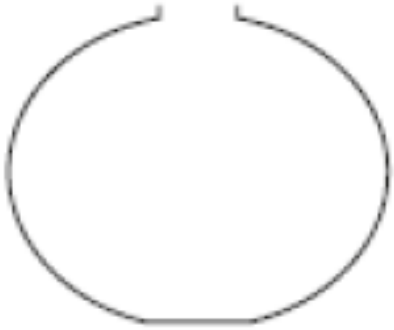

We can immediately exclude container A and B:
Container A's cross-section increases, therefore the graph should become **less steep over time**, which is the opposite of the graph shown
Container B's cross section initially increases, therefore in the first part of the graph, the curve should become less steep over time.

To decide between the last two containers, we notice that the last part of container C is cylindrical, i.e. it has a constant cross-section, and should therefore increase at a constant rate (it is a straight line), which is exactly what we observe happening in the graph.

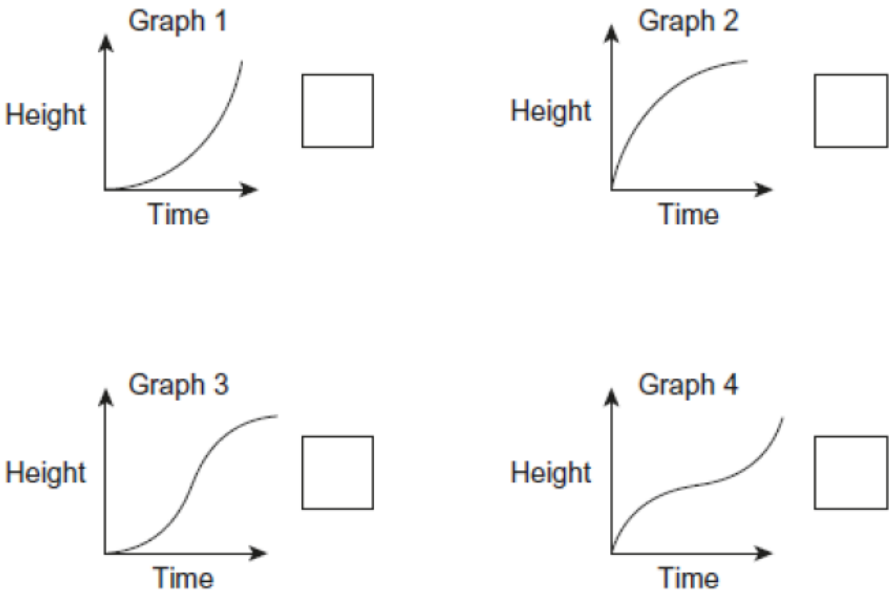
The answer is C.

**Example 2**

The following container is filled with water with a constant flow.



Which of the following graphs represents the container?




The cross-section of the container initially increases and then decreases: we expect the graph to have a decreasing steepness in the first half and becoming steeper and steeper in the second part. Graph 1 and Graph 2 can be excluded because their steepness only increases and only decreases respectively. Between Graph 3 and Graph 4 we choose the latter because the steepness initially decreases and increases in the second part.

The answer is Graph 4.

| Practice & Assessment |
| --- |
| Go to<br><br>https://student.desmos.com<br><br>and type in the class code on the board. |

# Tutorial 6 - Feedback

## What is the Purpose of Tutorial 7?

- To receive feedback on final assessments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.
- To reflect on the programme including what was enjoyed and what was challenging.

## Final assessment feedback

| What I did well… | What I could have improved on… |
|---|---|
| • | • |
| • | • |
| • | • |

| My target for future work is… |
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| |

## Reflecting on Uni Pathways

| What did you most enjoy about Uni Pathways? |
| --- |
| • <br> • <br> • |

| What did you find challenging about the programme? | How did you overcome these challenges? |
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# Notes

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# Notes

# Graphs: the subtle lines between Maths and real life

Key Stage 4 Programme

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