First Principles:

Understanding How Mathematics Works

Pupil Name	
Handbook Designed by	Dr. Watts

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1 (Baseline assessment)			
2 (Trigonometry and Geometry)			
3 (Limits and Derivatives)			
4 (Infinite Series)			
5 (Generating Functions)			
6 (Final assessment)			
7 (Feedback)			

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1	Baseline Assignment	
Tutorial 2	Assessments 1	
Tutorial 3	Assessments 2	
Tutorial 4	Assessments 3	
Tutorial 5	Assessments 4	

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Course Rationale

Mathematics is a beautiful system of logic that initially begins with several basic assumptions. We may then build on these assumptions (or axioms) to create a larger self-consistent system of statements or rules. This is what it means to derive results from first principles.

We often learn the rules of how to manipulate mathematics without fully understanding how these rules were originally derived. The true history of mathematics is the story of how these rules were discovered and how this knowledge continues to build.

This course will cover 4 interconnected areas of mathematics and seek to demonstrate how many of the rules that may seem natural and innate to us, are perhaps not truly so immediately obvious.

Do we really understand what the trigonometric functions represent? Do we know how their derivatives were determined? Do we know how to prove the link between trigonometry and exponential functions? Do we know about the more complex tools used to determine general terms of sequences?

These questions will be answered throughout this course, and hopefully provide a deeper insight into the inner workings of mathematics. Ultimately, mathematics is about discovery rather than the memorisation of rules, and this course will give us a glimpse of this principle at work.

Subject Vocabulary

Mathematical Term	Mathematical Definition	Example
Limit		
Derivative		
Generating Function		
Taylor Series		
Maclaurin Series		

1. Baseline Assessment

1.1 Pre-SEF

Before starting the course, you will be required to take part in completing a Self Evaluation Form. This form will ask you a series of questions to gauge your confidence and opinions on various education related topics, and will serve as a foundation on which to judge the effectiveness of the course.

Your answers to the questions in the Pre-SEF will not influence your final grade of the course in any way, and should be answered as truthfully as possible.

1.2 Baseline Assessment

You will be provided with a series of technical questions to determine your current level of mathematical understanding. These questions will target specific knowledge areas that are related to the course content you will be covering.

It is expected that at the conclusion of the course, you would subsequently feel more comfortable answering a similar series of questions.

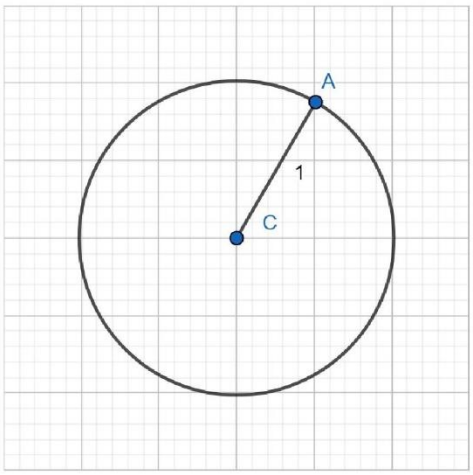
2. Trigonometry and Geometry

2.1 Objective

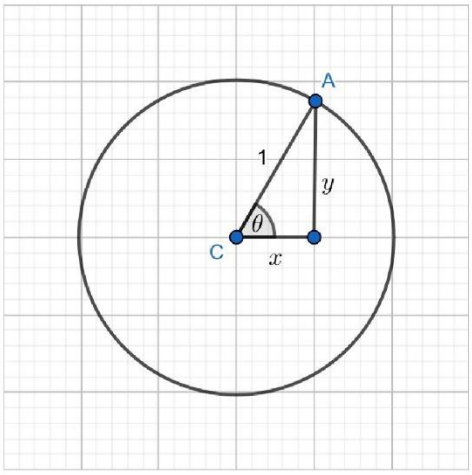
This section will explore the definition of the classic trigonometric functions from first principles. No assumptions will be made about their properties, instead we will derive these properties directly from the initial definition. Thus by the end of this tutorial, we should be capable of proving and re-deriving the trigonometric functions geometrically.

2.2 Practice: Forming a Unit Circle

We begin by drawing a circle with a centre at C and a radius of 1. We then mark an arbitrary point A on its circumference:



We now form a right angled triangle with the line segment between the centre of the circle C and the arbitrary point A acting as the hypotenuse. We also label the angle made by the two line segments meeting at the centre C as angle θ:



It should be clear that by moving the point A around the circumference of the circle, we would also be varying θ and subsequently obtain varying values of x and y, the horizontal and vertical sides of the triangle respectively. Let us now define functions that would allow us to calculate the values of the side lengths x and y, given the angle θ.

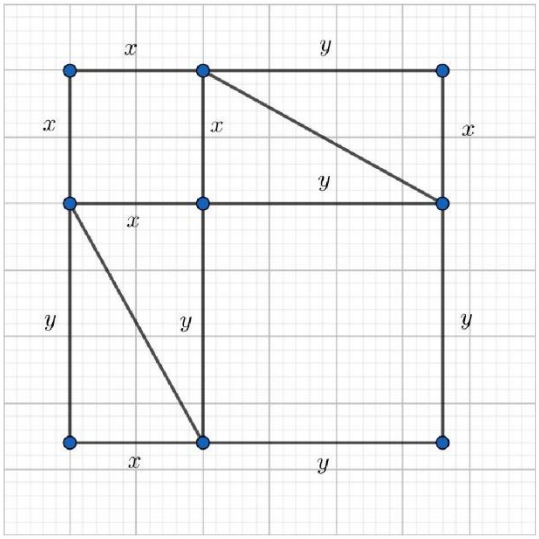
Definition 2.2.1 — Sine and Cosine. Given a right angled triangle with longest side of length 1 and some other angle θ we define the function $\sin(\theta)$ as the length of the side directly opposite the angle θ and $\cos(\theta)$ as the length of the side adjacent to the angle θ which is not the longest side.

Note that we will make no initial assumptions about our newly defined functions $\sin(\theta)$ and $\cos(\theta)$. Thus far we know only their definitions, and nothing yet about their properties or behaviour. However we can begin to derive further information about this pair of new functions by considering the geometric properties of our diagram.

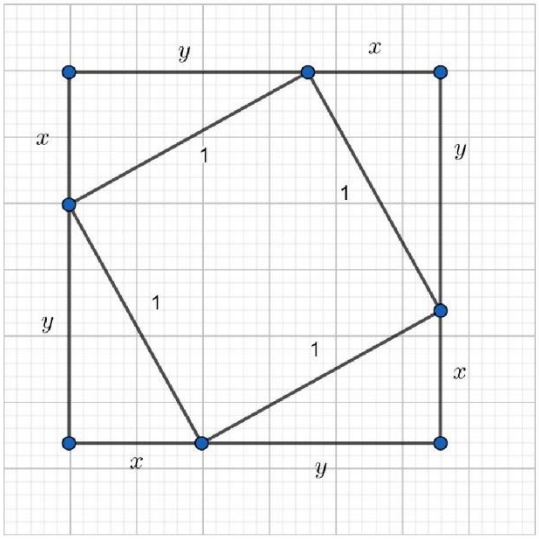
2.3 Practice: Rediscovering Pythagoras

Let us now consider our triangle with sides of length 1, x, y. We may make a geometrical arrangement of 4 instances of this triangle, in addition to two squares with side lengths x and y. Such an arrangement is shown in Arrangement 1 and has a total side length of $x+y$, thus a total area of $(x+y)^2$.
Alternatively, consider an arrangement using the same 4 triangles, combined with a single square of length 1. Such an arrangement is shown in Arrangement 2. Notice that once again the total side length and total area are $x+y$ and $(x+y)^2$ respectively.

Therefore by removing the 4 triangles from both arrangements, we may conclude that the remaining areas are identical. That is to say, that the areas of two squares with sides lengths x and y total to the area of the square of length 1. Thus we prove that $x^2 + y^2 = 1$.



Arrangement 1



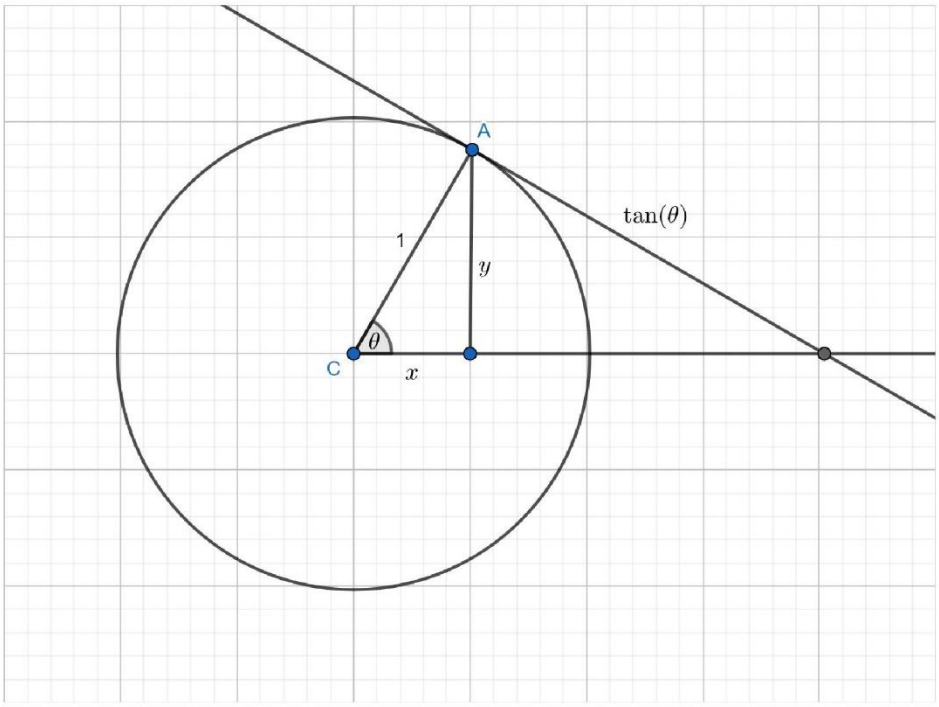
Arrangement 2

We have shown geometrically that $x^2 + y^2 = 1$, and since x and y are by definition $\sin(\theta)$ and $\cos(\theta)$, we have derived the Pythagorean Trigonometric Identity.

Theorem 2.3.1 — Pythagorean Trigonometric Identity. $\cos^2(\theta) + \sin^2(\theta) = 1$ for all θ

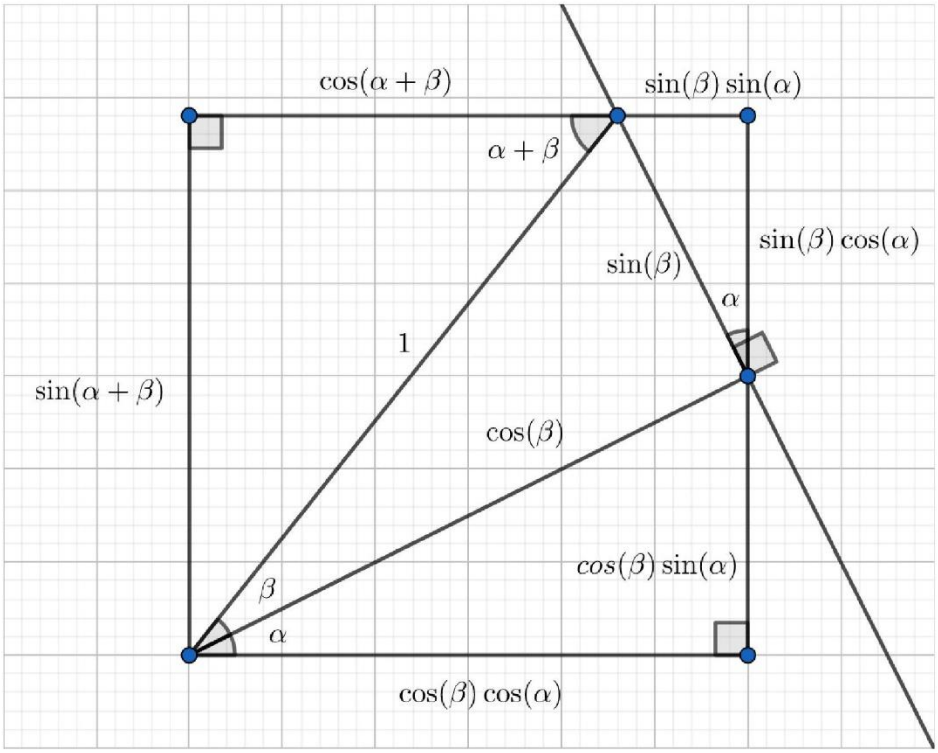
2.4 Practice: Deriving the Tangent Function

We now define the tangent function, appropriately named, as the length of the tangent to the circle starting from point A and connecting to the point at which the tangent crosses the horizontal passing through the centre C. It may subsequently be shown through the application of similar triangles that $\tan(\theta)$ is indeed equal to $\frac{\sin(\theta}{\cos(\theta)}$ as we might already suspect:



2.5 Doubling Angles

Consider the graphic on the following page, and the properties it demonstrates with regards to the pair of functions $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$. Incredibly, geometry is able to provide us the double angle formulas for trigonometry, and ultimately this is how they are originally derived. This single image captures a pair of trigonometric identities that are often assumed without justification.



Theorem 2.5.1 — Double Angle Identities.

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

2.6 Assessments

Assessment 2.1 Draw from memory the geometric diagram founded on the unit circle which shows the fundamental trigonometric functions as labels on the appropriate line segments. ■

Assessment 2.2 Draw from memory the geometric diagram which can be used to derive the double angles formulas for the sine and cosine functions. ■

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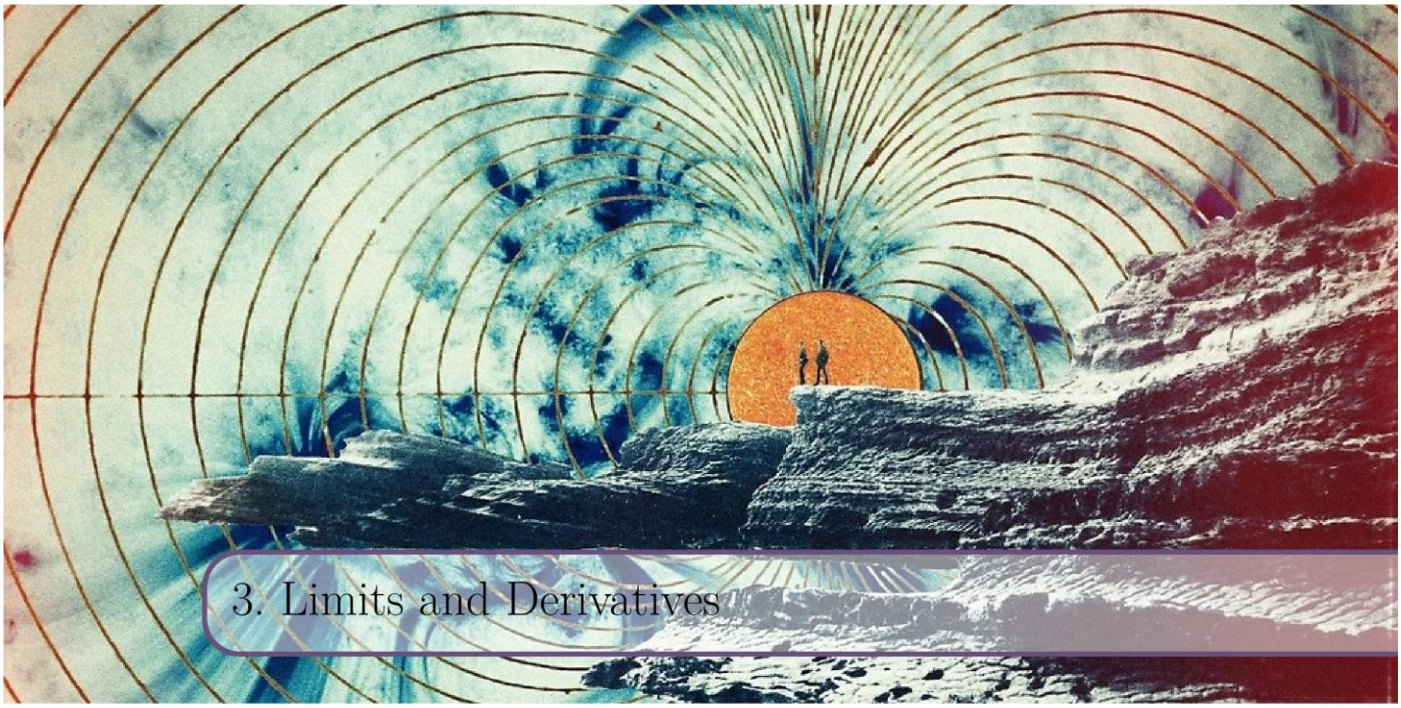
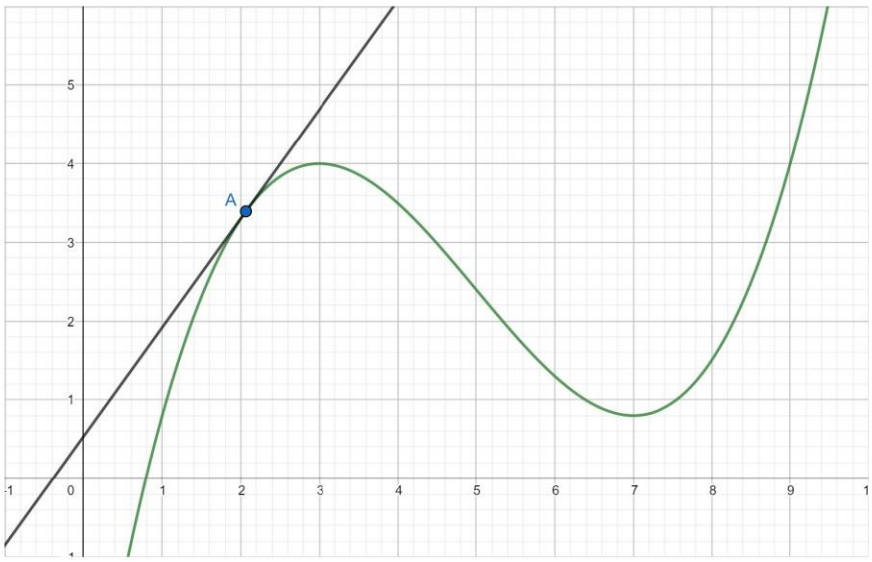


3. Limits and Derivatives

3.1 Objective

This section will explore the fundamentals of calculus, and the definition of derivatives. Using the properties of trigonometric functions that we derived in the previous tutorial we will determine how to differentiate them. In order to accomplish this, we will also introduce the concept of limits and understand how to take limits of common functions.

3.2 Practice: Defining the Derivative Informally



Let us consider for a moment what we mean by the derivative of a function $y = f(x)$. Without making reference to any further formulas, the derivative of a function at a specific point $x = a$, is defined as the instantaneous rate of change of y with respect to x when $x = a$.

This may be more commonly interpreted as the gradient of $f(x)$ when $x = a$, and can be determined by calculating the slope of the tangent line at the point where $x = a$. However if we wish to come up with a formal mathematical definition, we will need to make use of limits.

3.3 Practice: Experimenting with Limits

We make use of limits when we wish to consider the behaviour of a function as its input approaches a specific value, particularly when the function would ordinarily not be defined at that value.

To see this principle in action, consider the following function:

$$\frac{x^3 - 1}{x - 1}$$

When $x = 1$ it is clear that this results in a division by 0, and therefore the function is undefined at this value. However if we consider a sequence of values of x which tends towards $x = 1$ such as $1.1, 1.01, 1.001, 1.0001, \ldots$ we see that the corresponding output is $3.31, 3.0301, 3.003, 3.0003, \ldots$ respectively. Thus it appears the output tends towards some fixed value within the vicinity of 3.

Notice that if we factorise the function as shown below, we may make some progress in determining this mystery value:

$$\frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x - 1} = (x^2 + x + 1))$$

When the newly simplified function above is supplied with $x = 1$ we now clearly arrive at an output of 3, contrary to our previous attempts which resulted in an undefined result. We may summarise this result with the following notation:

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

A formal definition of this limit notation does exist, though this is not reproduced here. Instead, consider the examples below, and satisfy yourself that the values assigned to these limits may seem reasonable, when considering the neighbouring values:

$$\lim_{x\to 3} 4x = 12$$

$$\lim_{x\to 9} \frac{x}{3} = 3$$

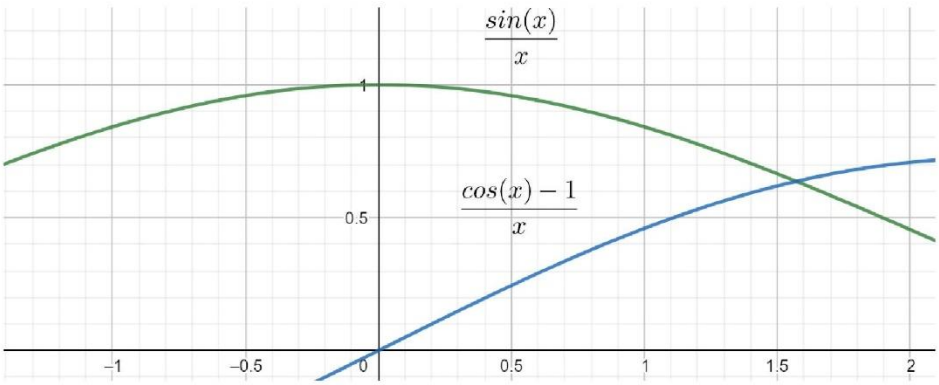
$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = 1$$

$$\lim_{x\to \infty} e^{-x} = 0$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

It proves beneficial to the understanding of limits, to use graphical software to plot these functions and visualise their behaviour when x tends towards its limiting value. This is particularly the case for the final two limits which include the sine and cosine functions:



3.4 Practice: Defining the Derivative Formally

Let us now define the derivative of a function formally, making use of this limit terminology. As discussed initially, the derivative of a function $y = f(x)$ at $x = a$ is merely the limit of the gradient of the tangent line as x tends towards a.

Definition 3.4.1 — Derivative. Given a function $y = f(x)$ we say that the derivative of the function at $x = a$ is $f'(a)$ where $f'(x)$ is defined:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3.5 Practice: Derivative of Sine

To showcase the power of limits, we may now derive the derivative of the sine function from first principles:

$$f(x) = \sin(x)$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} \sin(x) + \lim_{h \to 0} \frac{\sin(h)}{h} \cos(x)$$

$$= \sin(x) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \sin(x) \times 1 + \cos(x) \times 0$$

$$= \sin(x)$$

3.6 Practice: Derivative of Cosine

We may additionally determine the derivative of cosine, which demonstrates an interesting cyclical pattern within the derivatives of these two trigonometric functions:

$$f(x) = \cos(x)$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{(\cos(h) - 1)}{h}\cos(x) - \lim_{h\to 0} \frac{\sin(h)}{h}\sin(x) \\
&= \cos(x)\lim_{h\to 0} \frac{(\cos(h) - 1)}{h} - \sin(x)\lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \cos(x) \times 0 - \sin(x) \times 1 \\
&= -\sin(x)
\end{aligned}$$

3.7 Assessment

Assessment 3.1 Determine the limit: $\lim_{x\to 0} x\sin(1/x)$ ■

Assessment 3.2 Investigate why this limit does not exist: $\lim_{x\to 0} \sin(1/x)$ ■

Assessment 3.3 Use the limit definition to determine the derivative of $y = x^2$ ■

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4. Infinite Series

4.1 Objective

This section will explore the power of functions with infinite numbers of terms. We will learn the process by which we may characterise any function as an infinite polynomial. We will then apply this to the trigonometric functions, using the derivatives that we discovered in the previous tutorial. Finally, we aim to derive a fundamental formula of complex mathematics.

4.2 Practice: Forming a General Function

We know that all linear functions may be written in the form $mx+c$ and all quadratics are of the form ax^2+bx+c. However, it would be beneficial if we could take this process several steps further, and find a way of representing any function of x. Let us then assume that for any function $f(x)$ we might be able to represent this using the following infinite series:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_kx^k + \ldots$$

We have not proved that there is a set of constants a_i that would make this equality valid, or indeed if such a set of constants exists, that it would be unique. However, let us work with assumption, and experiment with the results it might bring.

4.3 Practice: Derivatives at Zero

Let us take repeated derivatives of this infinite sum:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_kx^k + \ldots$$

$$f^1(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + ka_kx^{k-1} + \ldots$$

$$f^2(x) = (2)(1)a_2 + (3)(2)a_3x + \cdots + (k)(k-1)a_kx^{k-2} + \ldots$$

$$f^3(x) = (3)(2)(1)a_3 + \cdots + (k)(k-1)(k-2)a_kx^{k-3} + \ldots$$

Now let us consider the derivative at $x = 0$ in each instance:

$$f(0) = a_0$$

$$f^1(0) = a_1$$

$$f^2(0) = (2)(1)a_2$$

$$f^3(0) = (3)(2)(1)a_3$$

Now the pattern becomes clear, and we may define the following relationship in general:

$$f^n(0) = n!\, a_n \iff a_n = \frac{f^n(0)}{n!}$$

So it seems we have determined a unique value for all the constants a_i which may be determined by taking successive derivatives of the function $f(x)$. Such an infinite polynomial expansion is known as a Taylor series. We may now continue this investigation further, by applying this principle to functions which may be differentiated indefinitely; namely the trigonometric functions.

4.4 Practice: Trigonometric Taylor Series

We know that $\sin(x)$ and $\cos(x)$ both have the following cycling sequence of derivatives:

$$\sin(x) \to \cos(x) \to -\sin(x) \to -\cos(x) \to \sin(x) \to \cos(x) \to \ldots$$

Therefore when $x = 0$ we respectively obtain the following cycling values:
$0 \to 1 \to 0 \to -1 \to 0 \to 1 \to \ldots$

If we now plug these values into our function that generates the polynomial constants a_i we may obtain a pair of infinite series for the sine and cosine functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Let us take note of these two series expansions, and turn our attention to the exponential function.

4.5 Practice: Exponential Taylor Series

The exponential function $f(x) = e^x$ is another function that lends itself well to repeated differentiation. In fact we know that this function is its own derivative, and therefore the derivative of any order when $x = 0$ will always be $e^0 = 1$. This gives us the appropriate expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots$$

4.6 Practice: Deriving Mathematical Beauty

Let us now take a direct route to derive a most beautiful result of mathematics. Consider now substituting the variable x with a constant multiple ix where $i = \sqrt{-1}$ is the imaginary constant. Let us also consider multiplying the sine function by the same imaginary constant. We leave the cosine function unchanged. We then obtain the following 3 expansions:

$$i\sin(x) = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots$$

Notice that the first two expansion directly sum to make the third. This then gives us an extremely valuable result.

Theorem 4.6.1 — Euler's Formula. $e^{ix} = \cos(x) + i\sin(x)$

Finally we may substitute the value $x = \pi$ into our new formula, and we obtain the fact that $e^{i\pi} = \cos(\pi) + i\sin(\pi) = (-1) + (0) = -1$.

Theorem 4.6.2 — Euler's Identity. $e^{i\pi} + 1 = 0$

4.7 Assessment

Assessment 4.1 Find the infinite series which represents: $\frac{1}{1-x}$ ■

Assessment 4.2 What is the limit of this infinite series: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ ■

Assessment 4.3 Prove that this infinite series has no limit: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ ■

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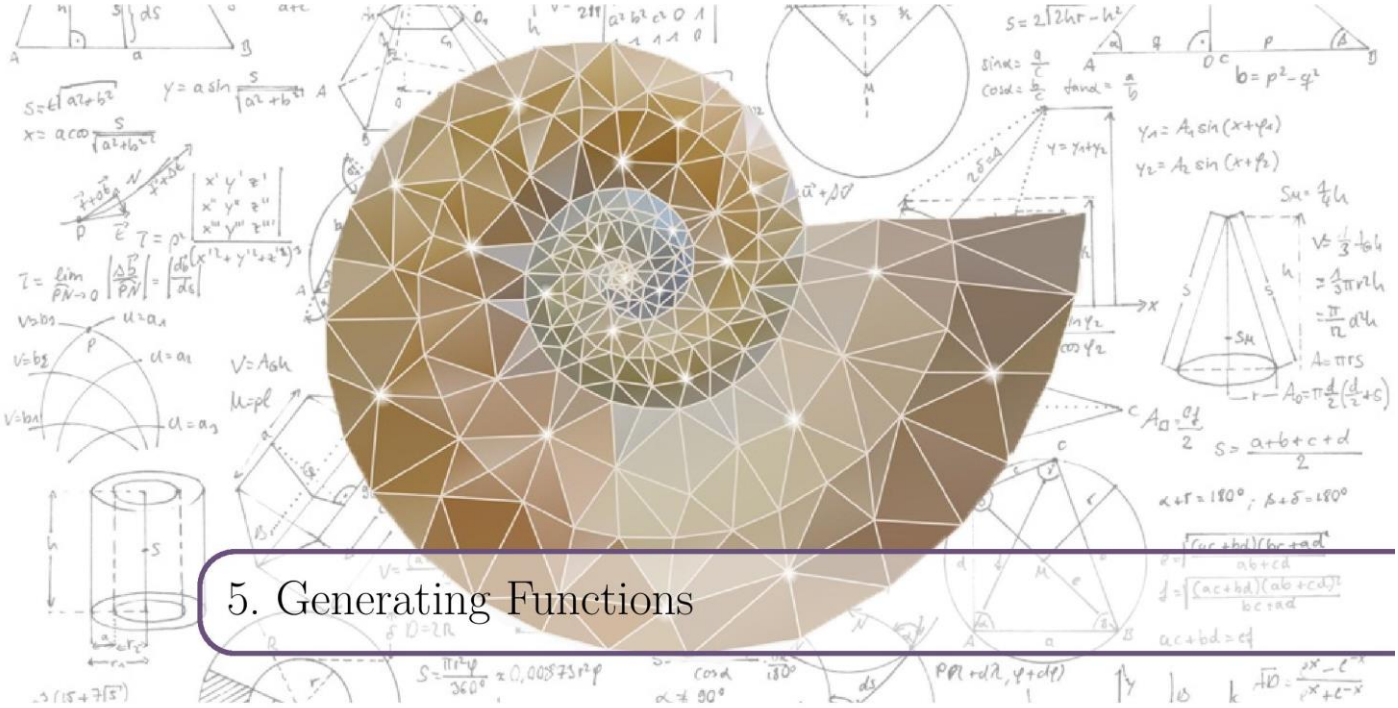


5. Generating Functions

5.1 Objective

This section will explore an extremely useful tool within mathematics, known as generating functions. These functions give us the power to morph a sequence across various domains, from sequence to polynomial to function. We will learn how to determine the generating function of a sequence, and ultimately learn the proof by which we can find any given term of the Fibonacci sequence.

5.2 Practice: From Sequences to Polynomials

Within mathematics, it often becomes useful to transform objects from one context to another, so that the tools of the alternative context may be used to manipulate the object. Often we may also transform the object back, carrying the newly found information with us. To take this abstract idea and provide a concrete example, we will consider converting number sequences into polynomials. The rule for doing so is incredibly simple.

Definition 5.2.1 — Generating Function. Given a sequence of numbers $F(0), F(1), F(2), F(3), \ldots$ we define the equivalent polynomial (known as a generating function) of the sequence as $G(x) = F(0) + F(1)x + F(2)x^2 + F(3)x^3 + \ldots$.

Let us consider a few examples to make this process clear:

$$0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots \longleftrightarrow x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

$$1, 1, 1, 1, 1, \ldots \longleftrightarrow 1 + x + x^2 + x^3 + x^4 + \ldots$$

$$1, -1, 1, -1, 1, \ldots \longleftrightarrow 1 - x + x^2 - x^3 + x^4 + \ldots$$

$$1, 0, 0, 0, 0, \ldots \longleftrightarrow 1$$

$$0, 0, 1, 0, 0, \ldots \longleftrightarrow x^2$$

$$1, 0, 1, 0, 1, \ldots \longleftrightarrow 1 + x^2 + x^4 + \ldots$$

5.3 Practice: From Polynomials to Generating Functions

For an infinite sequence, we obtain an infinite polynomial. However, we should now be aware that infinite polynomials may sometimes be represented in terms of simpler functions or algebraic expressions. Thus we may sometimes encapsulate an infinite sequence within a single simple mathematical function. This is why we use the term generating function to refer to such an object. We can see examples of these polynomials being collapsed below:

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \longleftrightarrow \log(1+x)$$

$$1 + x + x^2 + x^3 + x^4 + \ldots \longleftrightarrow \frac{1}{1-x}$$

$$1 - x + x^2 - x^3 + x^4 + \ldots \longleftrightarrow \frac{1}{1+x}$$

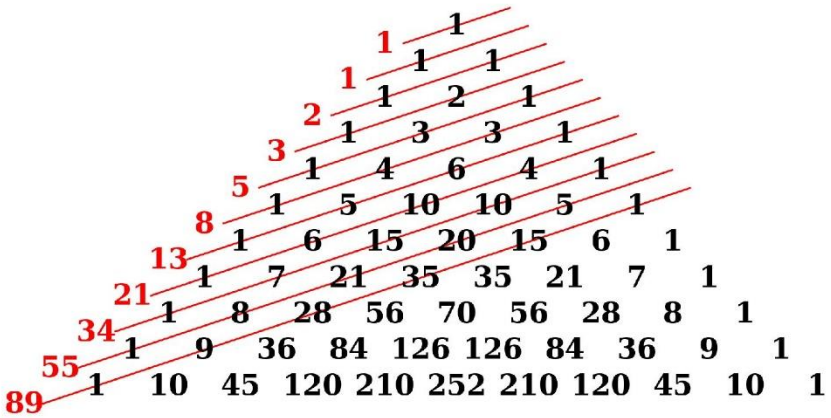
$$1 \longleftrightarrow 1$$

$$x^2 \longleftrightarrow x^2$$

$$1 + x^2 + x^4 + \ldots \longleftrightarrow \frac{1}{1-x^2}$$

5.4 Practice: Fibonacci

Few numerical sequences gain the same attention as that of the Fibonacci sequence. The sequence itself has a simple recursive definition, yet describes a pattern of numbers that appears in various mathematical contexts. Pascal's triangle is but one of these occurrences.



Definition 5.4.1 — Fibonacci Sequence. Define the nth term of the Fibonacci sequence as $F(n)$ where the function F has the following properties:

$F(0) = 0$
$F(1) = 1$
$F(n) = F(n-1) + F(n-2) \quad n \geq 2$

5.5 Practice: Tackling the Fibonacci Sequence

With the new tool of generating functions at hand, we may now begin the endeavour of finding a closed form expression of $F(n)$ the nth term of the Fibonacci sequence, so that we may find the nth term directly rather than calculate n recursive steps. We begin with the generating function $G(x)$ of the Fibonacci sequence:

$$G(x) = F(0) + F(1)x + F(2)x^2 + F(3)x^3 + \cdots + F(n)x^n + \ldots$$

Let us now multiply this expression by x and x^2 to obtain two further expressions:

$$G(x) = F(0) + F(1)x + F(2)x^2 + F(3)x^3 + \cdots + F(n)x^n + \ldots$$

$$xG(x) = F(0)x + F(1)x^2 + F(2)x^3 + \cdots + F(n-1)x^n + \ldots$$

$$x^2G(x) = F(0)x^2 + F(1)x^3 + \cdots + F(n-2)x^n + \ldots$$

We may now apply a clever addition and subtraction of these expressions that lines up with the Fibonacci recursive formula. Specifically we may subtract the 2 new expressions from our original expression:

$$G(x) - xG(x) - x^2G(x) = F(0) + F(1)x - F(0)$$

$$\implies (1 - x - x^2)G(x) = x$$

$$\implies G(x) = \frac{x}{1 - x - x^2}$$

We now have a clean expression of the generating function of the Fibonacci sequence. However this does not yet tell us the nth term of the sequence for any given n. In order to accomplish this, we need to expand into an infinite sequence once again. This will be left as an exercise. However the ultimate conclusion of this process is a surprising looking formula.

Theorem 5.5.1 — Fibonacci Formula.

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

5.6 Assessment

Assessment 5.1 Complete the proof of the derivation of the nth term of the Fibonacci sequence. ■

Assessment 5.2 Find the nth term of the function $F(n)$ where $F(0) = 0, F(1) = 1$ and with the recursive property that $F(n) = F(n-1) + 2F(n-2)$. ■

Assessment 5.3 Find 3 examples of $\phi = \frac{1+\sqrt{5}}{2}$ occurring in mathematical contexts or contexts in nature. Be wary of false claims. ■

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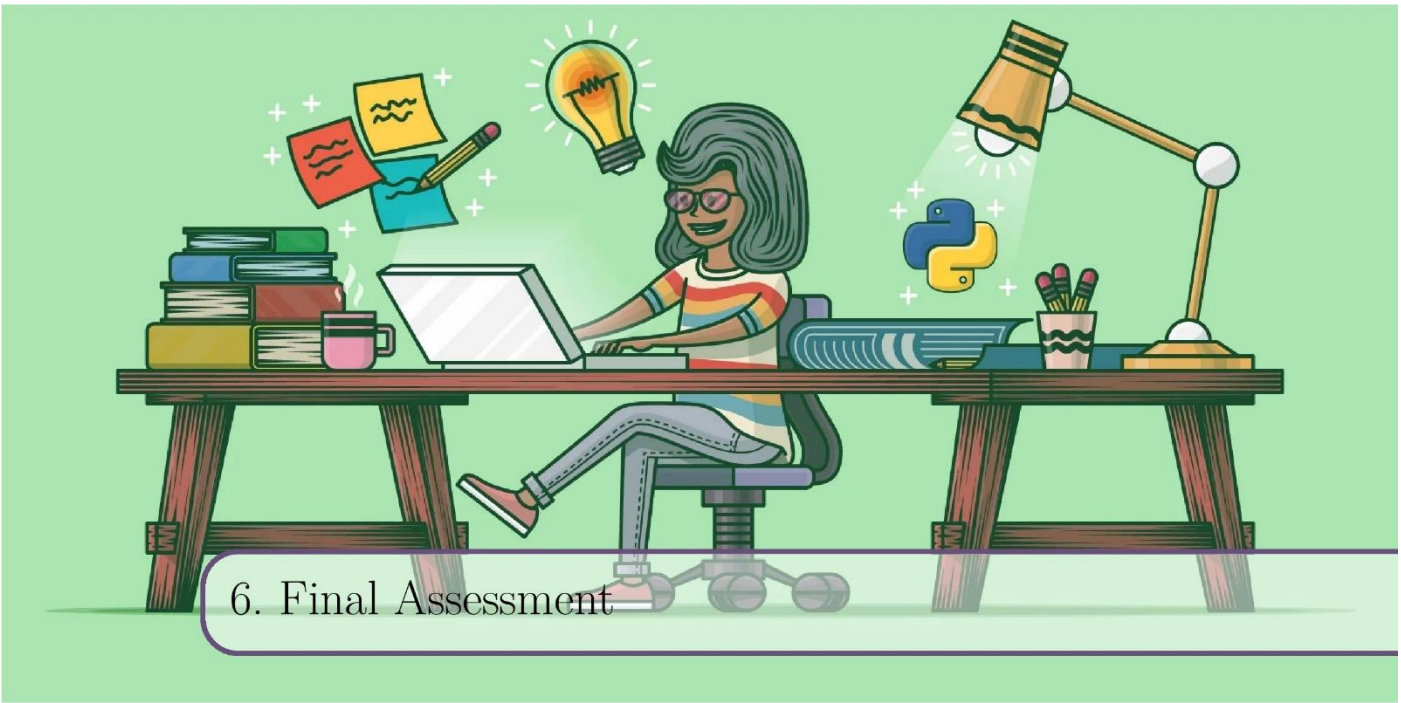

6. Final Assessment

6.1 Final Assessment

To complement the baseline assessment you took at the start of the course, you will now be given a final assessment in the form of a final project. This will allow you to review the content that you have covered throughout the course and elements of your own, driven by your own research in related mathematical topics of interest.

This assessment will give you the opportunity to demonstrate how far you have come in your knowledge acquisition, and provide an opportunity to produce an item of mathematical writing that expresses your personal interests. Your project will be formally assessed against a common set of criteria used by all schools participating with the Brilliant Club, and will form the bulk of your final grade at the end of the course. Good luck!

Tutorial 7 - Feedback

What is the Purpose of Tutorial 7?

- To receive feedback on final assessments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.
- To reflect on the programme including what was enjoyed and what was challenging.

Final assessment feedback

What I did well…	What I could have improved on…
•	•
•	•
•	•

My target for future work is…

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?
•   •   •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

UNI PATHWAYS

First Principles:

Understanding How Mathematics Works

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