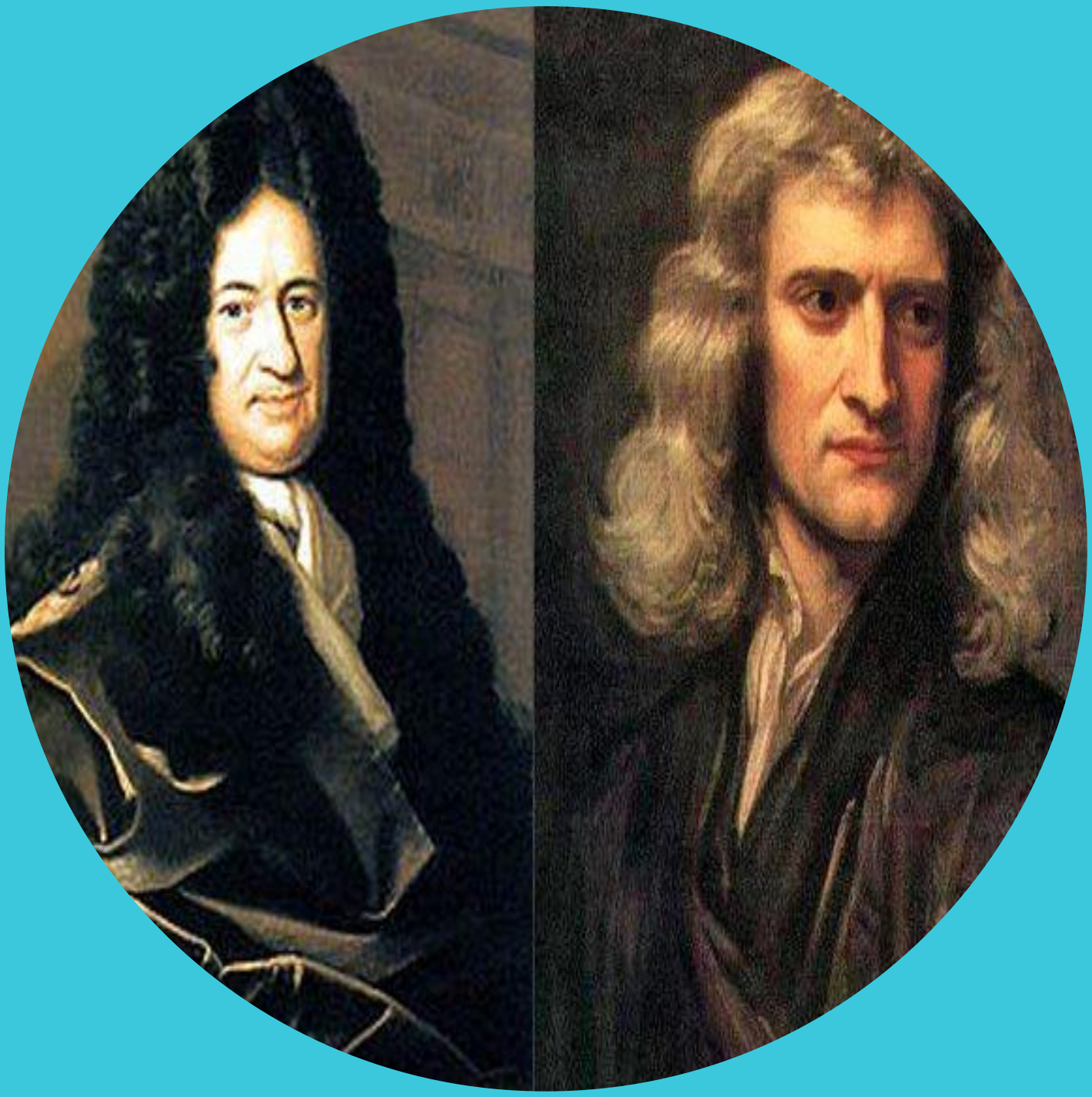

Integration

Key Stage 5 Revision Programme

Pupil Name

Handbook Designed by Dr. Danny Scarponi

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1 (Baseline assessment)			
2			
3			
4			
5			
6 (Final assessment)			
7 (Feedback)			

Timetable – Homework Assignments

Tutorial 1		
Tutorial 2		
Tutorial 3		
Tutorial 4		
Tutorial 5		

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Course Rationale

Integration is one of the two main operations of calculus, with its inverse operation, differentiation, being the other. Given a function f of a real variable x and an interval $[a, b]$ of the real line, the definite integral

$$\int_a^b f(x)\,dx$$

is defined informally as the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. The area above the x-axis adds to the total and that below the x-axis subtracts from the total.

The principles of integration were formulated independently by Isaac Newton and Gottfried Wilhelm Leibniz in the late 17th century, who thought of the integral as an infinite sum of rectangles of infinitesimal width. Bernhard Riemann gave a rigorous mathematical definition of integrals.

Integrals are used extensively in many areas of mathematics as well as in many other areas that rely on mathematics. For example, in probability theory, integrals are used to determine the probability of some random variable falling within a certain range. Integrals can be used for computing the area of a two-dimensional region that has a curved boundary, as well as computing the volume of a three-dimensional object that has a curved boundary. The area of a two-dimensional region can be calculated using the aforementioned definite integral.

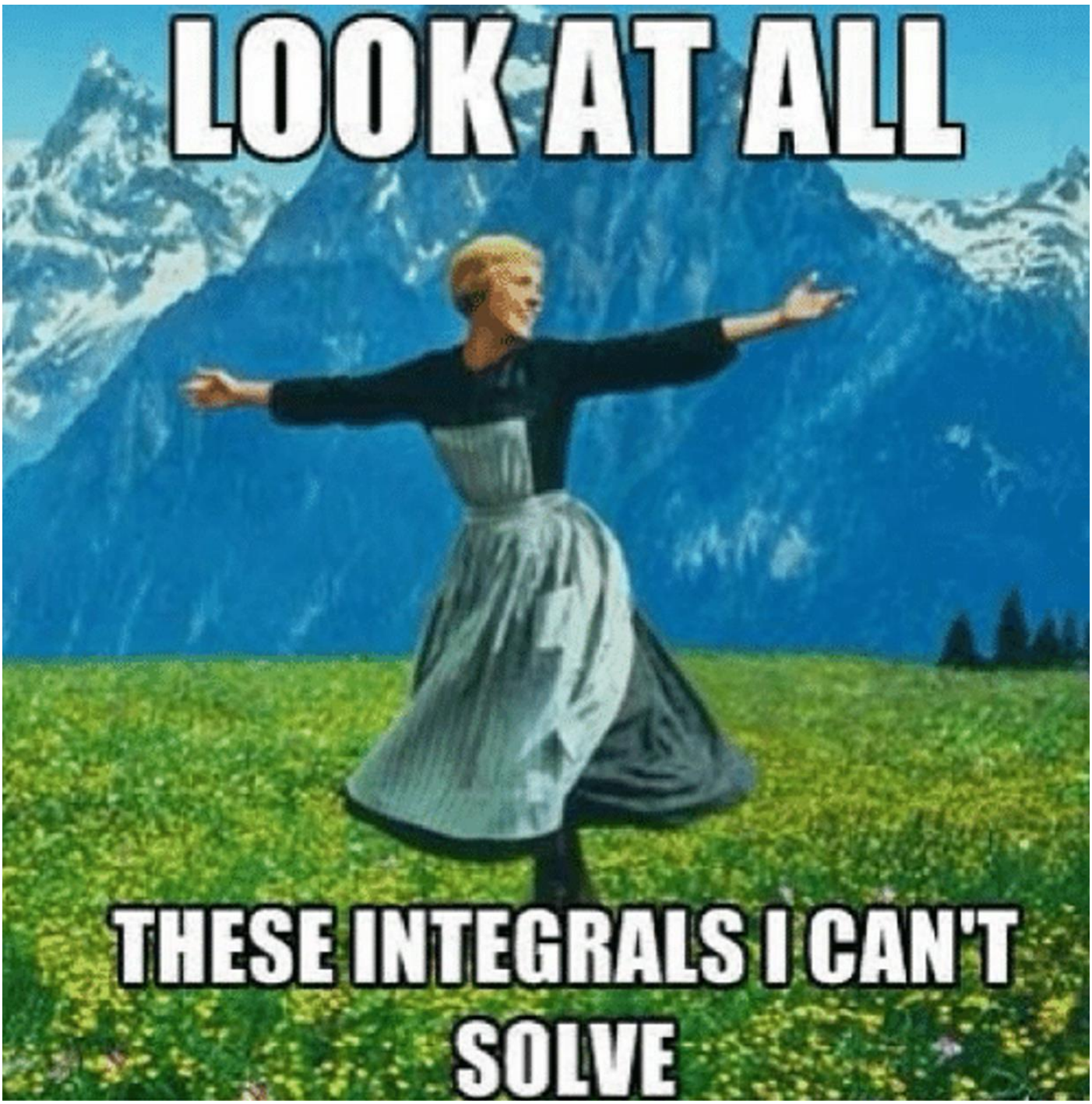
Mastering the art of integration is therefore essential for all those studying a scientific discipline at university (to cite a few: mathematics, physics, engineering, finance, and economics). The aim of this Unipathways course is to help students become more confident when dealing with integrals. There are a myriad of tools and techniques that are used to solve integrals. This can cause students difficulty, because it is not necessarily easy to know which technique to use in any given case.

The Unipathways course will focus on revising these various techniques and on helping students understand the strengths and weaknesses of each of them. Students will not be given any miraculous recipe (simply because such a recipe does not exist, and this is what makes mathematics so beautiful!), but they will learn to think in terms of what is more or less likely to work in one context rather than another.

Subject Vocabulary

Word	Definition	In a sentence
Indefinite integral		
Definite integral		
Rational function		
Partial fractions		
Trigonometric identity		
Integration by substitution		
Integration by parts		

Tutorial 1 – Baseline Assessment

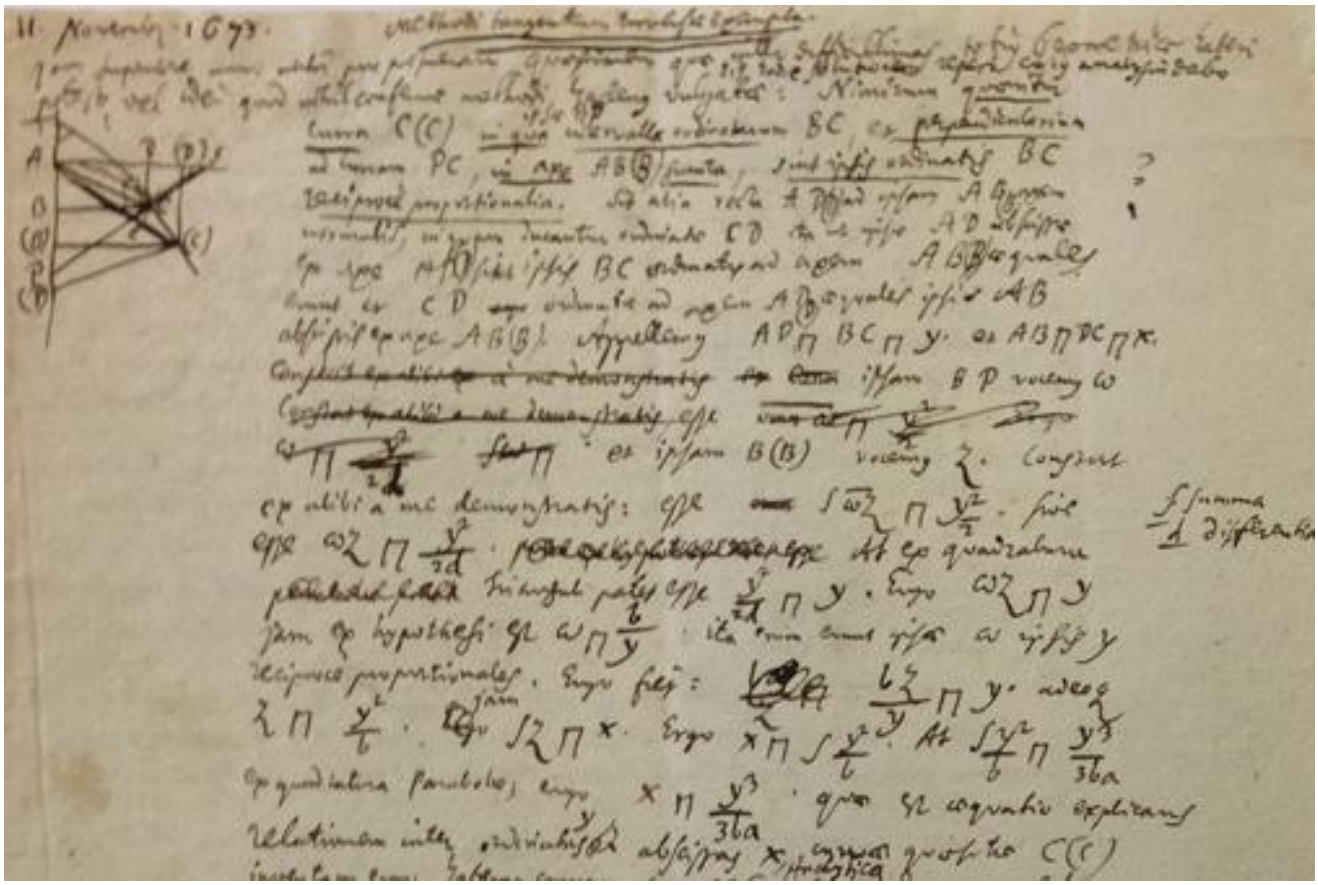


What is the Purpose of Tutorial 1?

- Pre-SEF
- Baseline assessment

Baseline Assessment

Tutorial 2 – Standard integrals



Leibniz's most famous notations are his integral sign (long "s" for "summa") and d, here summarized in the margin for the first time, on November 11th, 1675.

What is the Purpose of Tutorial 2?

- Objective A: Revise standard integrals and their generalizations
- Objective B: Use trigonometric identities to reduce certain integrals to standard ones

Objective A: Revise standard integrals and their generalizations

Complete the following table.

Derivative	Integral
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx =$
$\frac{d}{dx}\left(\sin x\right) = \cos x$	$\int cosx dx =$
$\frac{d}{dx}\left(\cos x\right) =$	$\int sinx dx =$
$\frac{d}{dx}\left(\tan x\right) =$	

$\frac{d}{dx}\left(\mathbf{e}^{x}\right)=$	
$\frac{d}{dx}\left(\ln\lvert x\rvert\right)=\frac{1}{x},\quad x\neq 0$	
$\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$	
$\frac{d}{dx}\left(\csc x\right)=$	
$\frac{d}{dx}\left(\cot x\right)=-\csc^{2} x$	

Any of the derivatives/integrals above can be generalized by replacing x with ax+b.

Example. The chain rule gives us

$$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}a$$

Therefore

$$\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

Following the example above, fill the table below.

Derivative	Integral
$\frac{d}{dx}cos(ax+b)=$	$\int sin(ax+b)=$
$\frac{d}{dx}sin(ax+b)=$	$\int cos(ax+b)=$
	$\int \frac{1}{ax+b}=$
	$\int e^{ax+b}=$
	$\int sec(ax+b)tan(ax+b)=$

Practice A

Find the following indefinite integrals

$\int (2x + 1)dx$

$\int (-5x^4 + 32x^3 - 39x^2 + 12x)dx$

$\int \left(\frac{-9}{40x^{\frac{7}{4}}}\right)dx$

$\int (2x-5)^6 dx$

$\int (-11x^{-1})dx$

$\int \frac{4}{2x+1} dx$

$\int \pi e^{5x-4} dx =$

$\int (-5\cos 2x)dx$

$\int sin(7x+4)dx$

$\int \left(\frac{1}{\sec -4x}\right)dx$

$\int (8\sec^2 5x)dx$

Assessment A

What is the right answer?

$$\int e^{5x}\left(\frac{e^{2x}}{7} + \frac{3}{e^{3x}}\right) dx$$

$A.\ \frac{1}{7}e^{7x} + 3e^{3x} + c$
$B.\ 7e^{7x} + \frac{3}{2}e^{2x} + c$
$C.\ \frac{1}{49}e^{7x} + 3e^{5x} + c$
$D.\ \frac{1}{49}e^{7x} + \frac{3}{2}e^{2x} + c$

$$\int cosec^2(3x-5)dx$$

$A.\ \frac{1}{3}cot(3x-5)+c$
$B.\ -\frac{1}{3}cot(3x-5)+c$
$C.\ \frac{1}{3}cot(5-3x)+c$
$D.\ 3cot(3x-5)+c$

Objective B: Use trigonometric identities to reduce certain integrals to standard ones

Sometimes we can use trigonometric identities to transform integrals into standard forms.

Use the identity

$$sin^2x + cos^2x \equiv 1$$

to derive complete the identities below.

$$tan^2x + 1 \equiv$$

$$1 + cot^2x \equiv$$

Complete the duplication formulas below

$$sin2x \equiv$$

$$cos2x \equiv$$

Example:

$$\int tan^2xdx = \int(sec^2x-1)dx = \int sec^2xdx - \int 1dx = tanx - x + c$$

Practice B

Find the following integrals

$$\int cos^2 x dx =$$

$$\int cos^2 7x dx =$$

$$\int cot^2 x dx =$$

$$\int sin^2 3x dx =$$

Challenging:

$$\int \frac{4cos2x}{sin^2 2x} dx =$$

Assessment B

What is the right answer?

$$\int (sinx + cosx)^2 dx =$$

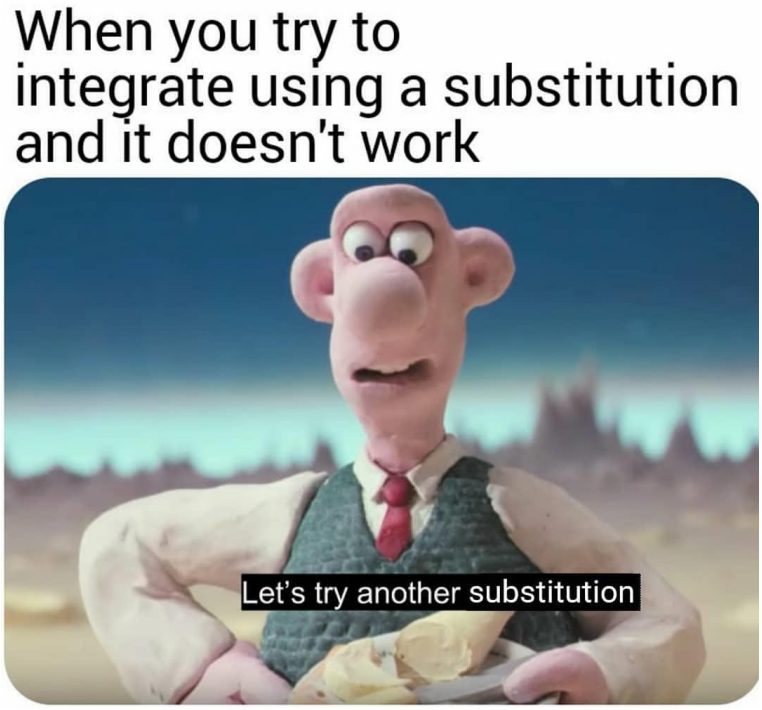
$A.\ 2x + cos2x + c$
$B.\ x + \frac{cosx}{2} + c$
$C.\ x - \frac{cos2x}{2} + c$
$D.\ \frac{x}{2} - cosx + c$

Challenge:

$$\int \frac{sinx}{1+sinx} dx =$$

Hint: Multiply the integrand by $\frac{1-sinx}{1-sinx}$

Tutorial 3 – Integration by substitution and partial fractions



What is the Purpose of Tutorial 3?

- Objective A: Integration by substitution
- Objective B: Use partial fractions to reduce certain integrals to standard ones

Objective A: Integration by substitution

Let us revise integration by substitution using an example.

Suppose we want to find the following integral: $\int x^2\sqrt{x^3+1}dx$.

We note that x^2 , which multiplies $\sqrt{x^3+1}$, is the derivative of x^3+1, up to a constant. This suggests that we must use the substitution $u = x^3+1$.

We have $\frac{du}{dx} = 3x^2$, or equivalently $dx = \frac{du}{3x^2}$. Therefore, we get

$$\int x^2\sqrt{x^3+1}dx = \int x^2\sqrt{u}\frac{du}{3x^2} = \frac{1}{3}\int\sqrt{u}du = \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{9}u^{\frac{3}{2}} + c$$

Practice A

Find the following integrals using an appropriate substitution.

$$\int tanxdx =$$

$$\int \frac{x^3}{(2-x^4)^5}dx =$$

$$\int \frac{cos\sqrt{x}}{\sqrt{x}}dx =$$

$$\int x^7 e^{x^8-2}dx =$$

$$\int \frac{e^x}{e^x+1}dx$$

$$\int \frac{ln(x^2+x^4)}{6x+12x^3}dx =$$

Challenge:

$$\int \frac{\sqrt{1-x}+3}{\sqrt{1-x}+1}dx$$

$$\int \frac{x^3}{\sqrt{9-x^2}}dx =$$

$$\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}dx =$$

$$\int \frac{\tan\left(\frac{4}{3}x\right)}{\cos^2\left(\frac{4}{3}x\right)}dx$$

Assessment A

What is the correct answer?

$$\int -4cotxdx =$$

$A.\ ln(|sin4x|) + c$
$B.\ -4ln(|sinx|) + c$
$C.\ -ln(4|sinx|) + c$
$D.\ -4ln(|cosx|) + c$

$$\int \frac{1}{xln(2x)}dx =$$

$A.\ 2ln(|ln|2x||) + c$
$B.\ ln(2|ln|x||) + c$
$C.\ ln(2|ln|2x||) + c$
$D.\ ln(|ln|2x||) + c$

Challenge:

$$\int \sin^3(x)\cos^2(x)dx =$$

Objective B: Use partial fractions to reduce certain integrals to standard ones

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where both $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$.

Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible.

Then for each factor in the denominator we can use the

following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in denominator	**Term in partial fraction decomposition**
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}, k = 1, 2, 3, \ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$

We know how to integrate $\frac{A_k}{(ax+b)^k}$ for $k = 1, 2, 3, \ldots$ (see previous tutorial).

In the third case, we need to integrate a term of the type $\frac{Ax+B}{ax^2+bx+c}$.

Here we will treat only the subcase $\frac{2ax+b}{ax^2+bx+c}$, i.e. the case $A = 2a$ and $B = b$.

In this case, the numerator is the derivative of the denominator, so we have an integral of the form $\int \frac{f'(x)}{f(x)} dx$.
We have already seen how to solve this:

$$\int \frac{2ax+b}{ax^2+bx+c} = ln(|ax^2+bx+c|) + c$$

Example:

$$\int \frac{9x^2-14x-5}{3x^3-7x^2-5x-3}dx$$

Step 1: Factorise the denominator.

Since 3 is a root of the denominator, then (x-3) is one of his factor. To find the oher factor(s) we can

divide the denominator by (x-3):

$3x^3 - 7x^2 - 5x - 3 = (x-3)(3x^2+2x+1)$

where $(3x^2+2x+1)$ cannot be factorised further (why?).

Step 2: Write the integrand as a sum of easy pieces

$\frac{9x^2-14x-5}{3x^3-7x^2-5x-3} = \frac{A}{x-3} + \frac{Bx+C}{3x^2+2x+1}$

Here A=1, B=6, C=2.

Step 3: Integrate the various pieces

$$\int \frac{9x^2-14x-5}{3x^3-7x^2-5x-3}dx = \int (\frac{1}{x-3} + \frac{6x+2}{3x^2+2x+1})dx = ln(|x-3|) + ln(|3x^2+2x+1|) + c$$

Practice B

Find the following integrals.

$$\int \frac{1}{x^2-4}\,dx$$

$$\int \frac{2x+3}{x^2-9}\,dx$$

$$\int \frac{4x-11}{x^3-9x^2}\,dx$$

Challenge:

$$\int \frac{3x^3-5x^2-11x+9}{x^2-2x-3}\,dx$$

Assessment B

What is the right answer?

$$\int \frac{8x^2-3x-4}{(4x-1)(x^2+1)}\,dx$$

$A.\ ln(|4x-1|) + 3ln(|x^2+1|) + c$

$B.\ -ln(|4x-1|) - 3ln(|x^2+1|) + c$

$C.\ -ln(|4x-1|) + \frac{3}{2}ln(|x^2+1|) + c$

$D.\ -ln(|4x-1|) - \frac{3}{2}ln(|x^2+1|) + c$

Challenge:

$$\int \frac{4x^3 + 2x^2 + 1}{4x^3 - x}\, dx$$

Tutorial 4 – Integration by parts

What is the Purpose of Tutorial 4?

- Objective A: Integration by parts: basic cases
- Objective B: Integration by parts: advanced cases

Objective A: Integration by parts: basic cases

Let us start by recalling the rule:

$$\int u(x)v'(x)\,dx \;=\; u(x)v(x) - \int u'(x)v(x)\,dx$$

Example: suppose we want to find $\int xe^x dx$.
Here we have two functions: x and e^x. We know how to integrate and how to differentiate both of them, but when we integrate x we get a more complex function, while if we integrate e^x it stays the same. This suggests that we set
$u = x$ and $\frac{dv}{dx} = e^x$, so that

$$\int xe^x dx = xe^x - \int e^x = xe^x - e^x + c$$

Practice A

Find the following integrals.

$$\int x^2 \ln(x) dx$$

$$\int_6^0 (2+5x)\, \mathbf{e}^{\frac{1}{3}x}\, dx$$

$$\int x \ln(x)$$

$$\int \sqrt{x^3} \ln\left(\sqrt[3]{x}\right) dx$$

$$\int \log(x^2+3) dx$$

$$\int 9t\sec^2(2t)\ dt$$

$$\int_1^3 (2-x)^2 \ln(4x)\, dx$$

Challenge:

$$\int x^{13} e^{(x^7)}\ dx$$

$$\int x^3 \sin x^2 dx$$

Assessment A

Choose the right answer.

$$\int x \sin(x) \cos(x) dx$$

$A.\ \frac{1}{4}(-xsin2x + sin2x) + c$
$B.\ \frac{1}{2}(-xcos2x + sin2x) + c$
$C.\ \frac{1}{4}(-xsin2x + \frac{sin2x}{2}) + c$
$D.\ \frac{1}{4}(-xcos2x + \frac{sin2x}{2}) + c$

Objective B: Integration by parts: advanced cases

Let us now see some more complex examples of integration by parts.

Example 1.

$$\int \ln(x)\,dx$$

In this case, the integrand does not immediately appear to be a product of two functions. Looking at it closer, we realise that we can interpret lnx as $lnx = 1 \cdot lnx$. Since we know how to integrate 1 and how to differentiate lnx,
we can use integration by part with.....

$$\int lnxdx =$$

Example 2.

$$\int x^2 \sin x\,dx$$

Here we can set $u = x^2 \text{ and } \frac{dv}{dx} = sinx$, obtaining

$$\int x^2 sinxdx = -x^2 cosx + 2\int xcosx$$

At this point, we need to integrate by parts again....

Example 3. In some cases, we can use integration by parts to get back to the original integral and then solve an equation. Let's see how this works. Suppose you are given the following integral: $\int e^x cosxdx$.
We can set $u = cosx \text{ and } \frac{dv}{dx} = e^x$ to get

$$\int e^x cosxdx = e^x cosx + \int e^x sinx$$

We can integrate by parts again, setting $u = sinx \text{ and } \frac{dv}{dx} = e^x$, to get

$$\int e^x cosxdx = e^x cosx + \int e^x sinxdx = e^x cosx + e^x sinx - \int e^x cosxdx$$

Rearranging the equation, we get

$$2\int e^x cosxdx = e^x cosx + e^x sinx + c$$

or equivalently

$$\int e^x cosxdx = \frac{e^x cosx + e^x sinx}{2} + c$$

Practice B

Find the following integrals.

$$\int e^{2\theta}\sin(3\theta)d\theta$$

$$\int x\frac{\cos(x)}{\sin^3(x)}dx$$

$$\int \log^2(x)dx$$

$$\int_3^4 (x-1)\ln(x^2-2)dx$$

$$\int x^3\cos(x)dx$$

Challenge:

$$\int_0^\pi \sin nx \sin mx \ (n \neq m)dx$$

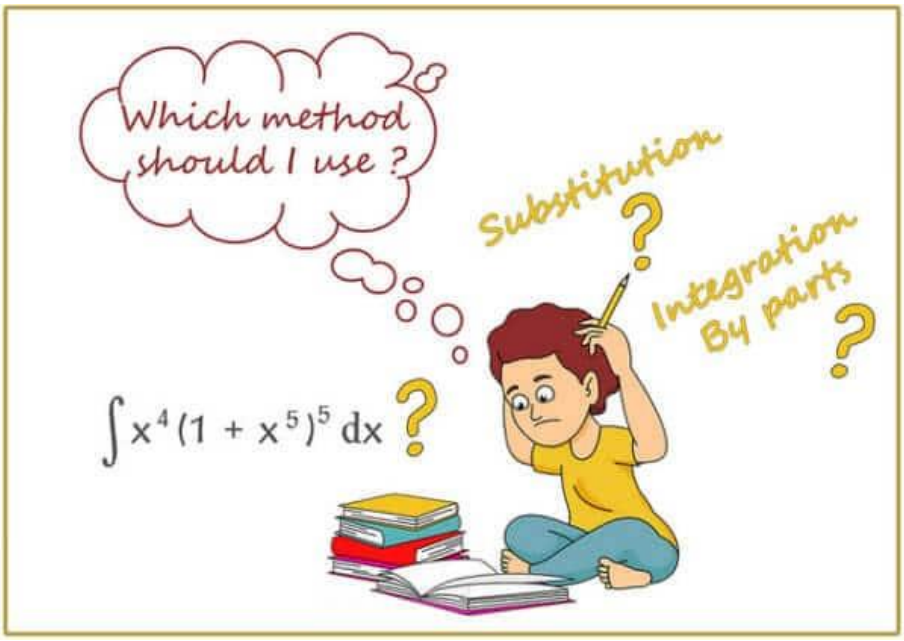
Assessment B

Choose the right answer.

$$\int sin(lnx)dx$$

$A.\ \frac{x}{2}(sin(lnx) - cos(lnx)) + c$
$B.\ \frac{x}{4}(sin(lnx) + cos(lnx)) + c$
$C.\ \frac{x}{2}(-sin(lnx) + cos(lnx)) + c$
$D.\ x(sin(lnx) - cos(lnx)) + c$

Tutorial 5 – Choosing the right integration method



What is the Purpose of Tutorial 5?

- Objective A: How to solve a given integral when no hint is given
- Objective B: Integrals and areas

Objective A: How to solve a given integral when no hint is given

When you need to find an integral and no hint is given, ask yourself the following questions (in order)

1) Is it a standard integral?
2) Is it a linear transformation of a standard integral?
3) Is there a trigonometric identity that transforms the integral in a standard integral?
4) Is it a rational function? In this case you can divide the numerator by the denominator (if possible) and then use partial fractions
5) Is there a substitution that makes the integral easier to deal with?
6) Is the integrand a product of two functions so that I can use integration by parts?

Important remark: in general, you may need to use more than one technique of integration to find an integral. Also, there is often more than one possible approach. Let us consider the following example.

Example: $\int e^{2x} ln(1+e^x)dx$

Here it is easy to exclude points 1)-4) above. We can try point 5) with the substitution $e^x = t$ (both e^{2x} and $ln(1+e^x)$ are functions of e^x).
Since $e^x dx = dt$, we get

$$\int e^{2x} ln(1+e^x)dx = \int t^2 ln(1+t)\tfrac{1}{t}du = \int t ln(1+t)dt.$$

The integral in the t variable seems slightly better than the one we had at the beginning. Now we can go through the points above again. It seems clear now that 1)-5) are not useful. We try to integrate by parts. It is easy to differentiate $ln(1+t)$, so we choose $u = ln(1+t)$ and $\frac{dv}{dx} = t$ and we obtain

$$\int tln(1+t)dt = \frac{t^2}{2}ln(1+t) - \frac{1}{2}\int \frac{t^2}{1+t}dt.$$

Now we are reduced to find the integral of a rational function. Since the degree of the numerator is bigger than the degree of the denominator, we need to divide. We can use the following trick:

$$\int \frac{t^2}{1+t}dt = \int \frac{t^2-1}{t+1}dt + \int \frac{1}{t+1}dt = \int (t-1)dt + ln|t+1| = \frac{t^2}{2} + t + ln|t+1| + c$$

So,

$$\int tln(1+t)dt = \frac{t^2}{2}ln(1+t) - \frac{1}{2}(\frac{t^2}{2} - t + ln|t+1|) + c$$

and substituting $e^x = t$

$$\int e^{2x}ln(1+e^x)dx = \frac{e^{2x}}{2}ln(1+e^x) - \frac{1}{2}(\frac{e^{2x}}{2} - e^x + ln(e^x+1)) + c.$$

Practice A

Find the following integrals.

$$\int e^{\sqrt{3s+9}}ds$$

$$\int \frac{xe^x}{(1+x)^2}dx$$

$$\int \sqrt{1+\cos^2(x)}\sin(2x)\cos(2x)dx$$

$$\int \frac{\cot(x)}{\sqrt{\sin(x)+1}+1}dx$$

$$\int \sin(x)\ln(\sin(x))dx$$

Assessment A

Choose the correct answer.

$$\int \frac{1}{\tan(x)+\cos(x)}dx$$

$A.$ $\frac{1}{\sqrt{5}}\left(ln|sinx + \frac{1+\sqrt{5}}{2}| - ln|-sinx + \frac{1-\sqrt{5}}{2}|\right) + c$

$B.$ $-\frac{1}{\sqrt{5}}\left(ln|sinx - \frac{1+\sqrt{5}}{2}| - ln|-sinx + \frac{1-\sqrt{5}}{2}|\right) + c$

$C.$ $-\frac{1}{\sqrt{5}}\left(ln|-sinx - \frac{1+\sqrt{5}}{2}| - ln|-sinx + \frac{1-\sqrt{5}}{2}|\right) + c$

$D.$ $\frac{1}{\sqrt{5}}\left(ln|sinx - \frac{1+\sqrt{5}}{2}| - ln|sinx + \frac{1-\sqrt{5}}{2}|\right) + c$

Objective B: Integrals and areas

Some exercises on integrals present you with a graph and ask you to find a certain area. In this case you have to understand what the integrand is and what the extremes of integration are.

Example:

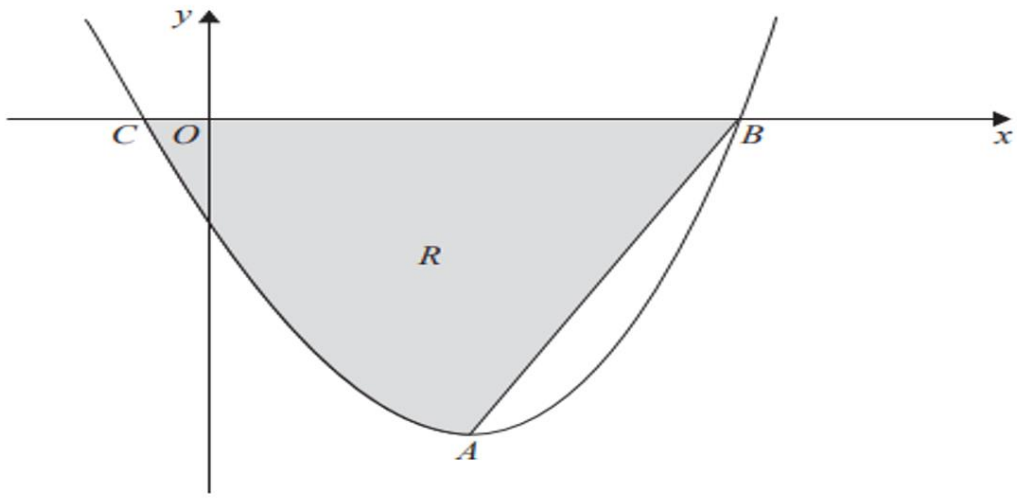


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = 4x^3 + 9x^2 - 30x - 8, \quad -0.5 \leqslant x \leqslant 2.2$$

The curve has a turning point at the point A.

(a) Using calculus, show that the x coordinate of A is 1

(3)

The curve crosses the x-axis at the points $B(2, 0)$ and $C\left(-\frac{1}{4}, 0\right)$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the line AB, and the x-axis.

(b) Use integration to find the area of the finite region R, giving your answer to 2 decimal places.

(7)

For (a) we must differentiate the curve y and find the values of x for which y' is equal to …

You should find that the point A has x-coordinate 1.

For (b) we can split the shaded region in two parts: from -1/4 to 1 and from 1 to 2. Therefore, we can write

Area of shaded region=

$$-\int_{-\frac{1}{4}}^{1}(x^3 + 9x^2 - 30x - 8)dx + \frac{(2-1)\cdot 25}{2}$$

=

$$[x^4 + 3x^3 - 15x^2 - 8x]_{-\frac{1}{4}}^{1} + 12.5 = 32.52$$

Practice B

1)

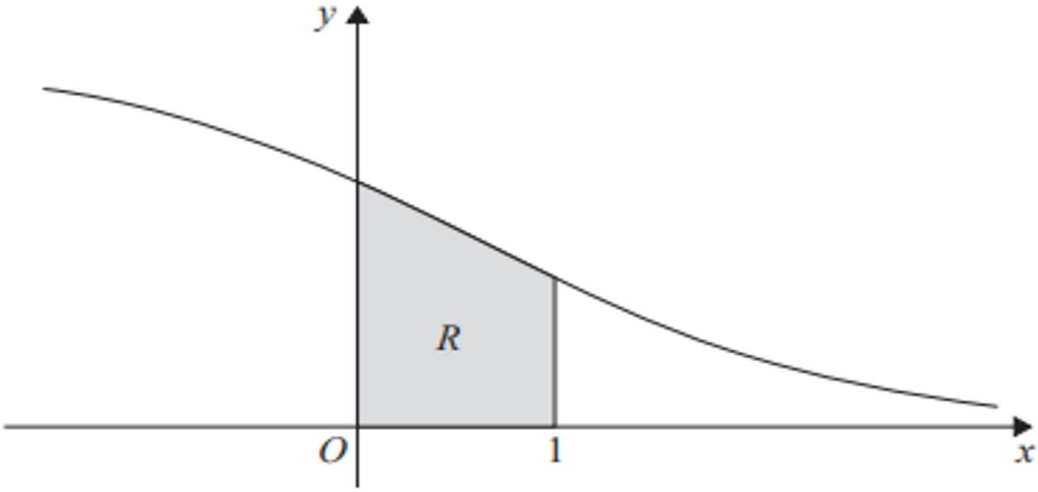


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{6}{(e^x + 2)}$, $x \in \mathbb{R}$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the y-axis, the x-axis and the line with equation $x = 1$

Find the area of the region R.

2)

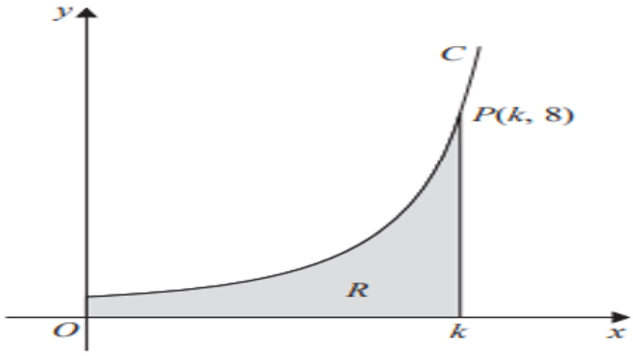


Diagram not drawn to scale

Figure 4

Figure 4 shows a sketch of part of the curve C with parametric equations

$$x = 3\theta \sin\theta, \qquad y = \sec^3\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point $P(k, 8)$ lies on C, where k is a constant.

(a) Find the exact value of k.

(2)

The finite region R, shown shaded in Figure 4, is bounded by the curve C, the y-axis, the x-axis and the line with equation $x = k$.

(b) Show that the area of R can be expressed in the form

$$\lambda \int_{\alpha}^{\beta} \left(\theta \sec^2\theta + \tan\theta \sec^2\theta\right) d\theta$$

where λ, α and β are constants to be determined.

(4)

(c) Hence use integration to find the exact value of the area of R.

(6)

Assessment B

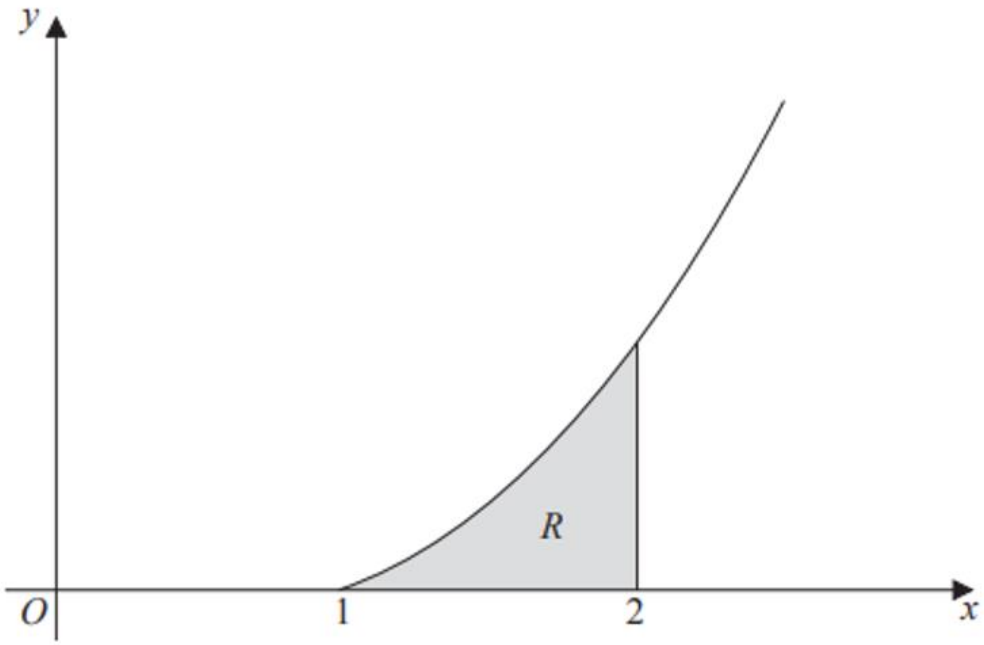


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = x^2 \ln x, \quad x \geqslant 1$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line $x = 2$

Find the area of the region R.

Tutorial 6 – Final Assessment

What is the Purpose of Tutorial 6?

- Post-SEF
- Final assessment

FInal Assessment

Tutorial 7 - Feedback

What is the Purpose of Tutorial 7?

- To receive feedback on final assessments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.
- To reflect on the programme including what was enjoyed and what was challenging.

Final assessment feedback

What I did well…	What I could have improved on…
•	•
•	•
•	•

My target for future work is…

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?
• • •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

Notes

Notes

Notes

Integration

Key Stage 5 Programme

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