A not so straight introduction to curves

Key Stage 4 Programme

Pupil Name	
Handbook Designed by	Jacopo Scalise

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1			
2			
3			
4			
5			
6 (Feedback)			
7 (Feedback)			

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1		
Tutorial 2		
Tutorial 3		
Tutorial 4		
Tutorial 5		

Assignment Submission – Lateness and Plagiarism

Lateness	
Submission after midnight on 9th August	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Uni Pathways – Pupil Feedback Report

Grade	Marks	Guidance
1st	70+	
2:1	60-69	
2:2	50-59	
3rd	40-49	
Working towards a pass	0-39	
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Name of PhD Tutor	Jacopo Vittorio Scalise		
Title of Assignment			
Name of Pupil			
Name of School			
ORIGINAL MARK / 100		FINAL MARK / 100	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the PhD tutor should give an explanation in this section:

Learning Feedback Comment 1 - ***Knowledge and understanding***	
What you did in relation to this Key Learning Priority	How you could improve in the future

Learning Feedback Comment 2 – ***Problem Solving***	
What you did in relation to this Key Learning Priority	How you could improve in the future
Learning Feedback Comment 3 – *Structure and presentation*	
What you did in relation to this Key Learning Priority	How you could improve in the future
Resilience Comment	
How you showed learning resilience during the course	How you could build learning resilience in the future

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Course Rationale

"I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and them classify them in order, is by recognizing the fact that all points of those curves which we may call "geometric," that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by a single equation."

René Descarted, La Geométrie, 1637.

This course is meant to be a very partial introduction to plane algebraic curves. These objects have attracted the attention of many mathematicians in the last 5 centuries. One of the main reasons of this fascination is that they provide an elementary, yet strikingly deep, connection between algebra and geometry.

What is algebra all about? When we think about algebra, we think about letters and numbers, equations and relations. What is geometry instead? The focus of geometry is on shapes and measures. From the 17th century on, the borders between these appearently distinct fields of mathematics have become more and more indistinct, as the use of coordinates in the Cartesian plane in order to solve geometric problems has become a standard for every mathematician. Today, we no longer distinguish between the equation $y=x+1$ and the line it underlies. So, if $y=x+1$ is a line, what kind of shape is $x^3+y^3=3xy$? It can be drawn in the plane but... it's not quite straight.

In the first tutorial we will mostly deal with numbers and letters, in order to have a clear vision of the formulas we will be drawing afterwards.

In the second tutorial we will go through a formal definition of algebraic curve and learn how to draw its graph with a software.

In the third tutorial we will learn how to modify the shape of a given curve, by moving it and stretching it on the plane.

In the fourth tutorial, we will study the relation between the shape of a curve and a certain whole number attached to curves, called degree.

In the fifth tutorial, we will study one of the most famous historical theorems about curves, which predicts in how many points two curves can meet.

Mark Scheme

STEM Mark Scheme

Skills	1st (70-100)	2:1 (60-69)	2:2 (50-59)
Knowledge and Understanding	• All content included is relevant to the general topic and to the specific question/title • Good understanding of all the relevant topics. • Scientific terms are defined and used accurately throughout • Clear justification on how the content included is related to the specific issues that are the focus of the assignment	• Most of the content included is relevant to the general topic and to the specific question/title • Good understanding of most the relevant topics • Scientific terms are used accurately but not always clearly defined. • Adequate justification on how the content included is related to the specific issues that are the focus of the assignment	• Some of the content included is relevant to the general topic and to the specific question/title • Good understanding on some of the relevant topics but occasional confusion on others. • Scientific terms are used mostly accurately with occasional confusion and often not defined. • Some justification on how the content included is related to the specific issues that are the focus of the assignment
Critical Evaluation / Problem solving	• Moved beyond description to an assessment of the value or significance of what is described • Evaluative points are consistently explicit/systematic/reasoned/justified • Effective critiques on the reliability of sources provided • Consistently demonstrate clear, analytical and logical steps to solving problems	• Mostly description but some assessment of the value or significance of what is described • Evaluative points are mostly explicit/systematic/reasoned/justified • Some evidence of critiques on the reliability of sources provided • Some examples of solving problems but not consistently clear, analytical and logical	• Only description with minimal assessment of the value or significance of what is described • Evaluative points are at times explicit/systematic/reasoned/justified • Limited evidence of critiques on the reliability of sources provided • Demonstrate clear steps to solving problems but not consistently analytical or logical

Structure and Presentation	• Ideas are presented in paragraphs and arranged in a logical structure that is appropriate for the assignment • The introduction clearly outlines how the essay/report will deal with the issues • The conclusion summarises all the main points clearly and concisely • All calculations, formulas and methods are clearly structured, clear to follow and correct • Tables and graphs are effectively constructed including appropriate headings, units and scales. • All sources are referenced correctly in an agreed format	• Ideas are presented in paragraphs and arranged in a structure that is mostly appropriate for the assignment • The introduction adequately describes how the essay/report will deal with the issues • The conclusion summarises most of the main points clearly • Calculations, formulas and methods are mostly structured, clear to follow and correct • Most tables and graphs are well constructed • Most sources are referenced correctly in an agreed format	• Ideas are presented in paragraphs and arranged in a structure • The introduction mentions how the essay/report will deal with the issues • The conclusion summarises some of the main points clearly • Calculations, formulas and methods are not always structured, clear to follow and correct. • Some tables and graphs are well constructed but contains some errors • Some sources are referenced correctly in the agreed format with occasional errors

Knowledge Organisers

Monomials and Polynomials in x, y

Monomial: an expression of the form $nx^i y^j$, where n (the coefficient) is any number and $i, j \geq 0$ (the exponents) are whole numbers.

Examples: $3x^2 y, -\frac{5}{2}x^5, 0.1x^3 y^3$

Polynomial: a sum of monomials that do not share the same exponents.

Example: $3x^2 y - \frac{5}{2}x^5 + 0.1x^3 y^3$

Degree: the degree of a monomial is the sum of its exponents. Examples: The degree of a polynomial is the greatest degree among the degrees of its monomials.

Examples: the degree of $0.1x^3 y^3$ is 3+3=6. The degree of $3x^2 y - \frac{5}{2}x^5 + 0.1x^3 y^3$ is the greatest among 2+1, 5+0 and 3+3, which is 6

Polynomial Equation: an equation whose terms are both polynomials. Most interesting for the course: polynomial equations whose second term is 0.

Examples: $\frac{9}{5}xy + 9 = 2x^2 y^6 - x$, $xy - 1 = 0$.

Plane algebraic curves and their main features

Plane algebraic curve: consists of all the points $(x; y)$ in the cartesian plane which satisfy a polynomial equation of the form $p(x, y) = 0$.

Example: for every point belonging to the curve with equation $xy - 1 = 0$, the multiplication of the abscissa with the ordinate equals 1. The points $(1;1), (2;\frac{1}{2}), (-\frac{1}{3};-3)$ ... all belong to this curve.

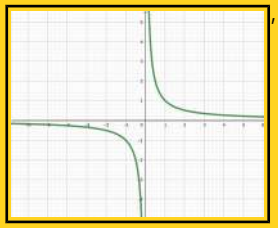
Components: the separated pieces of a curve.

Bounded/unbounded curve: a curve that can/cannot be framed in its entirety into a rectangle inside the Cartesian plane.

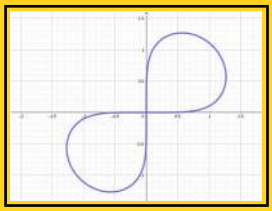
Singular point: a special point of a curve where it self-crosses or has a spike.

Singular/smooth curve: a curve with/without singular points.

Smooth, 2 components, unbounded



Singular, 1 component, bounded



Constructing and intersecting curves

Horizontal dilation: "stretching or compression" of a curve along the x-axis. It is obtained exchanging x for $\frac{x}{k}$ in the equation, where k is any number (called scale factor).

Vertical dilations: same as the horisontal dilations, but on on the y-axis.

Horisontal translation: horisontal movement of a curve which does not alter its shape or size. It is obtained exchanging x for $x - t$ in the equation, where t is any number (called step).

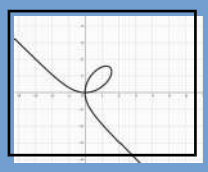
Vertical translation: same as the horisontal translations, but on on the y-axis.

Union of two curves: curve obtained by merging together two known curves. Its polynomial is the product of the polynomials of the two curves.

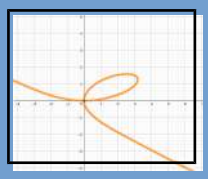
Intersection of two curves: a point in which 2 curves meet.

Bezout's theorem: the number of intersections two curves can have is smaller than the product of their degrees.

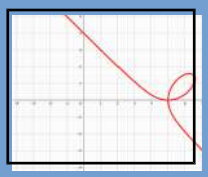
Original curve



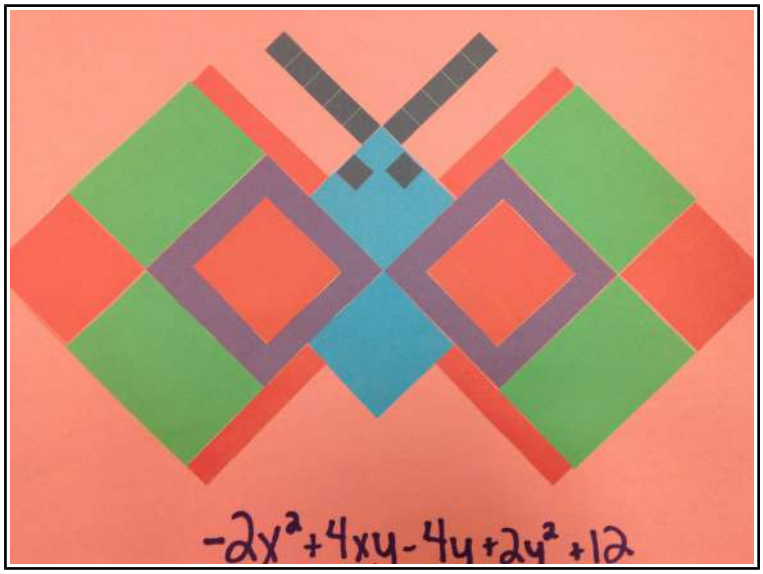
Dilated curve



Translated curve



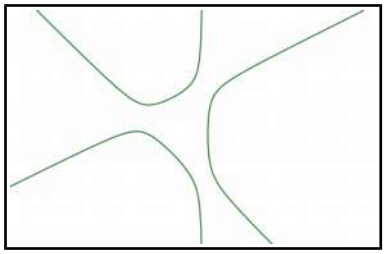
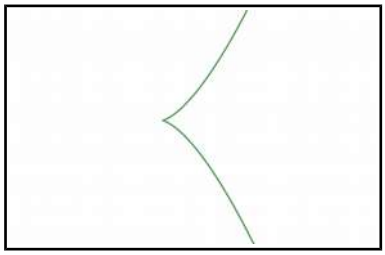
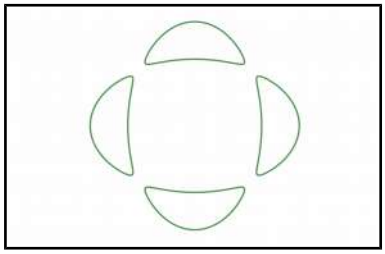
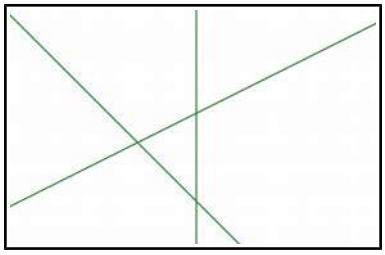
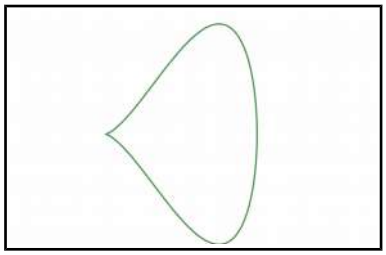
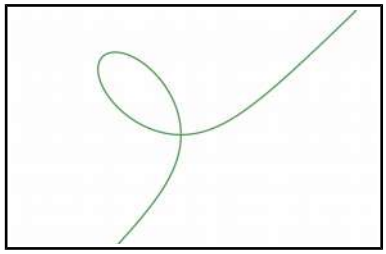
Tutorial 1 – Mono & Poly



What is the Purpose of Tutorial 1?

- To learn what a monomial and a polynomial are.
- To learn how to add and multiply polynomials.
- To learn the basic features of plane curves.

Task 1 – How would you describe the following images?

Complete the following table:

BOUNDED	SINGULAR POINT	SMOOTH	COMPONENTS

Key concept - Monomials in the unknowns x and y

A **monomial** in the unknowns x and y is an expression containing 3 objects:

1. a number, called coefficient ($2, -\frac{2}{3}, 0{,}23\ldots$)
2. a power of the unknown x ($x^2, x^4, x^1\ldots$)
3. a power of the unknown y ($y^3, y^5, y^{10}\ldots$)

For example: $2x^2y^3, 3xy^5, -\frac{1}{5}x^5y^{10}\ldots$

Careful: we admit 0 powers for the unknowns. This means, for example, that $y^3 = x^0y^3$ counts as a monomial in x and y.

Complete the following notes:

Two monomials are **similar** if...

_____

_____

For example...

_____

The sum of two similar monomials is....

_____

_____

For example...

_____

You can always multiply two monomials:

_____

_____

Task 2 – Quiz

You will see some sentences about monomials appearing on the board. For each one of them, you will have 2 minutes to discuss wether they are true or false and agree on an answer.

Key concept – Polynomials in the unknowns x and y

A polynomial in x and y is an expression consisting of a sum of non-similar monomials.
For example:

How to sum two polynomials (copy the example):

How to multiply two polynomials:

Task 3 – Polyrolling

Roll the dice twice. Pick two polynomials from the table according to the outcomes, and multiply them.
Repeat 3 times!

	⚀	⚁	⚂	⚃	⚄	⚅
Roll 1	y^2-x^2	$x-y-2$	y^3+x	x^4-3y	$3y$	$6x-y^4$
Roll 2	x^5+y^5-1	$-y^2-x^3$	$3xy^2-2$	$y^4+x^2y^2-1$	y^3-xy-7	$4xy^3-y+1$

Calculations

Baseline Assignment

The assignment is due next week.

1) Write down a short paragraph including the description of each of the curves in the first task, using the appropriate words you learned in the first tutorial.
Success criteria. The descriptions are clear and written in the appropriate language. The different curves are compared to highlight the better explain the meaning of the technical vocabulary you will use.

2) Calculate:

a) $(1-x)(1-xy)(1-y^2)$

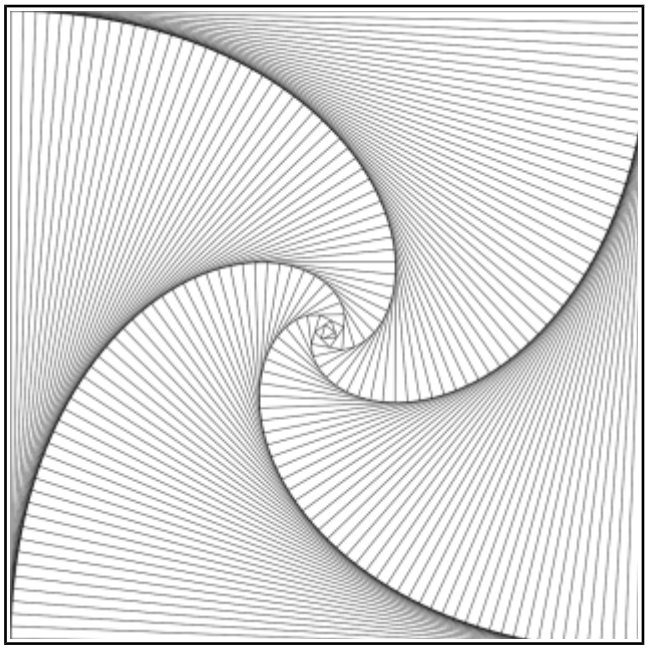
b) $y(x^2+y^2)-xy^5+y^3(xy^3-1)$

c) $(x^2-\frac{3}{7}y)14x^2+9xy(xy-3y+4x^3)$

d) $(0.2xy-0.4y^3)(-5xy+4y^3)$

Success criteria. All of your working is shown and the answers are correct.

Tutorial 2 – Fantastic curves and how to graph them



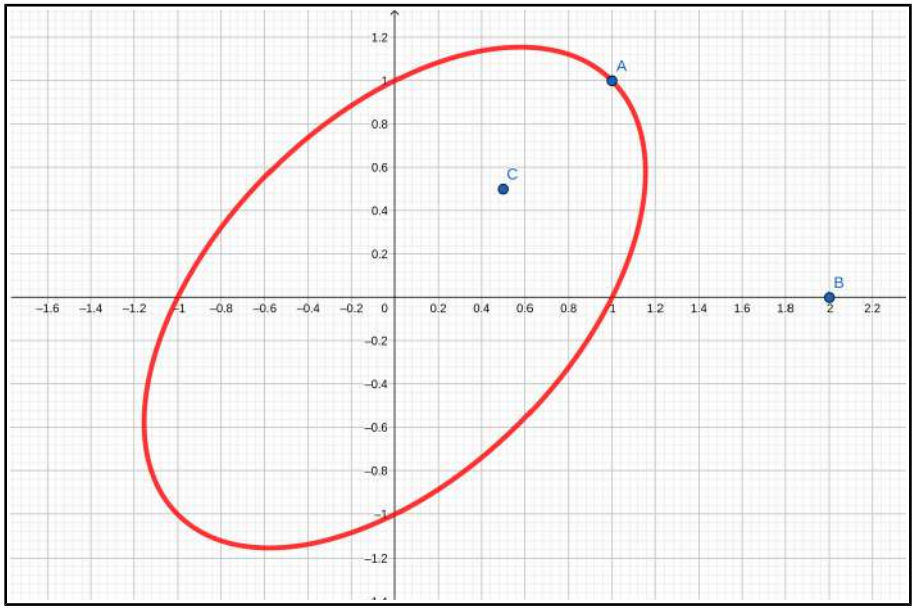
What is the Purpose of Tutorial 2?

- To understand the formal definition of algebraic plane curve.
- To be able to verify whether a point belongs to a curve.
- To be able to plot an algebraic plane curve using Geogebra.

Key concept – Plane algebraic curves

A plane algebraic curve is the space of points in the plane which make a certain polinomial "vanish". What does it mean?

An example: take the curve given by the equation $x^2+y^2-xy-1=0$. Here is how it looks like:



The point **A** with coordinates (1;1) belongs to the curve, because by substituting x=1 and y=1 in the equation, we find: $1^2+1^2-1\times1-1=0$. Remember: between two consecutive letters, we are secretly multiplying!

What happens if we try the same with the points **B**(2;0) and **C**(0.5;0.5)?

To sum up: **a plane algebraic curve is the set of solutions of an equation P=0**, where **P** is a polynomial in x and y.

Task 1 – Do I belong here?

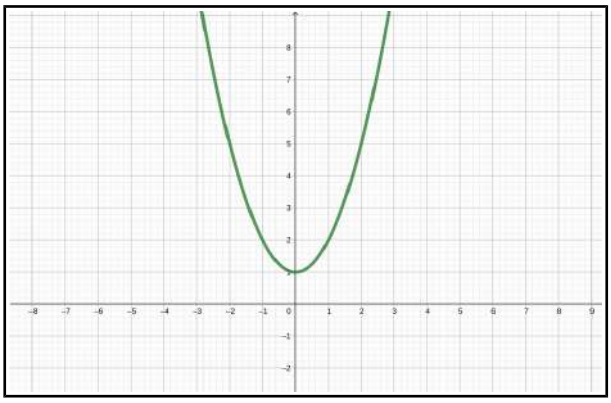
Each of the following points on the xy plane belongs to one and only one of the curves. Connect each point with the right curve!

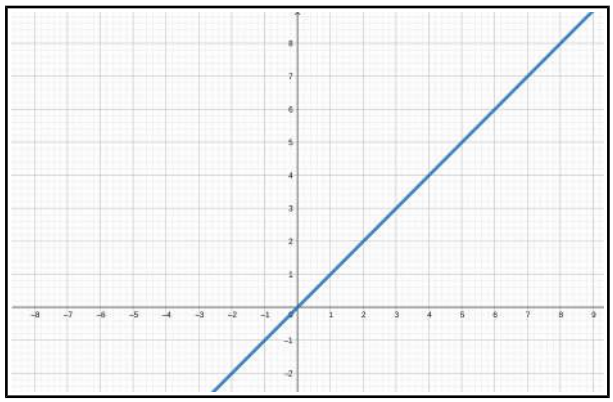
1) $y-2x+2=0$	a) (2;-3)
2) $x^3-y^2+1=0$	b) (10;0.5)
3) $xy-5=0$	c) (-4;-1)

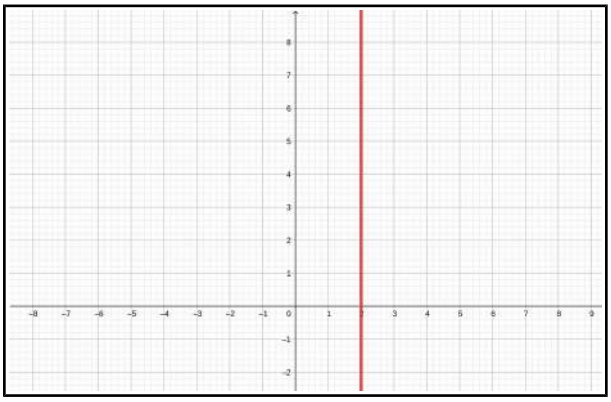
Task 2 – I recognise that old fellow!

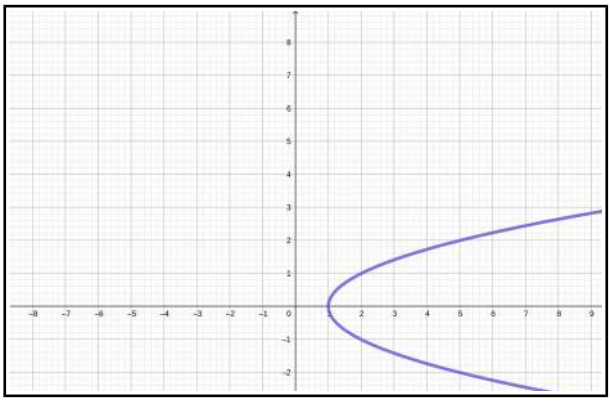
Can you assign each equation to the correct correspoinding image?

a) $y-x^2-1=0$
b) $x-2=0$
c) $x-y^2+1=0$
d) $y-x=0$









Task 3 – Geogebra

Plot the graph of the curves listed in Task 1.

Homework

1) Plot the following curves using Geogebra.

a) $x^3+y^3-3xy=0$

b) $y^2-0.8x^3=0$

c) $(x-2y)(x+y)(x-2)=0$

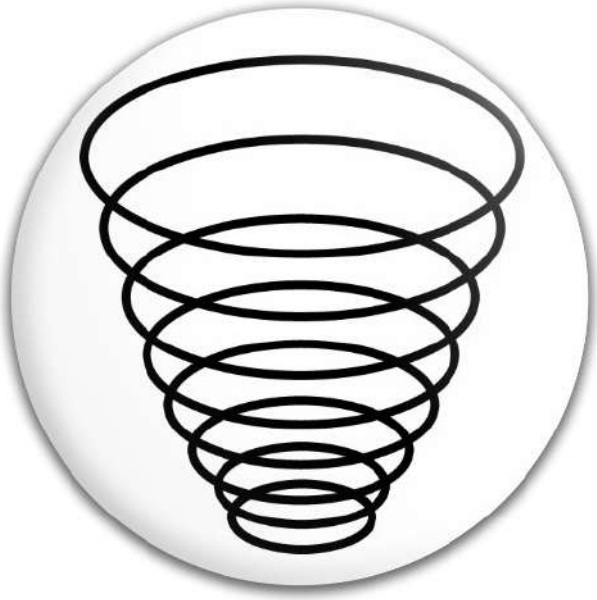
d) $0.4x^2+y^2-1=0$

e) $(0.4x^2+y^2-1)(x^2+0.4y^2-1)=0$

f) $(0.4x^2+y^2-1)(x^2+0.4y^2-1)+0.01=0$

2) Look at the curves d, e and f from the previous task. What do you notice? Write a short paragraph (max 200 words).

Tutorial 3 – Playing with curves: scalings, translations and unions

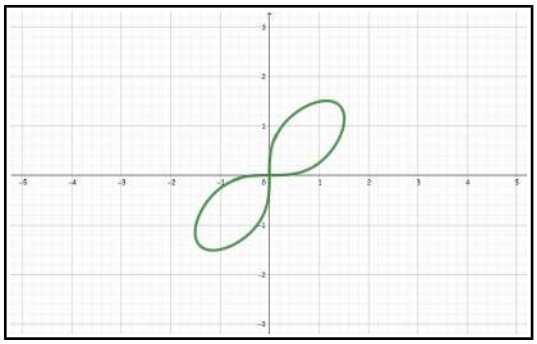


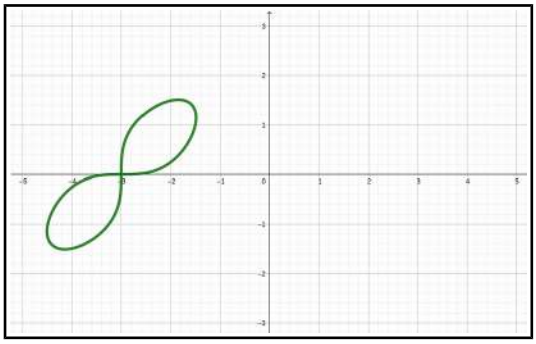
What is the Purpose of Tutorial 3?

- To know what a translation of a curve is.
- To know what a dilation of a curve is.
- To know how to find the equation of the union of two curves.

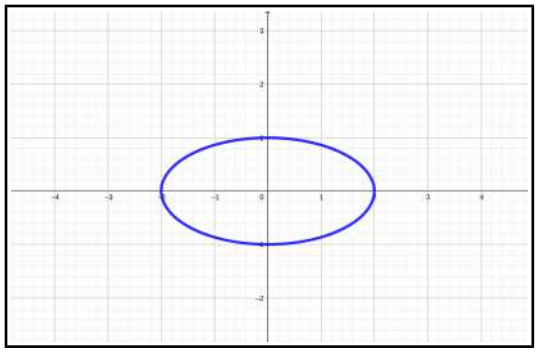
Task 1 – Can you explain the differences?

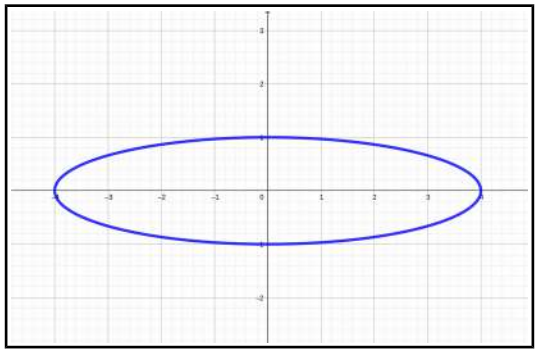
Discussion: look at the foolowing two curves. How would you describe the main difference between them?





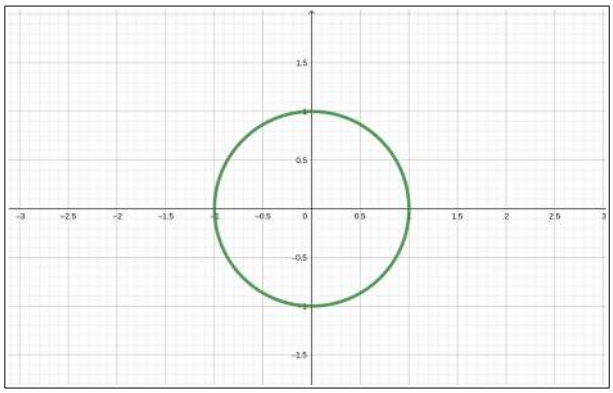
What about the next two?



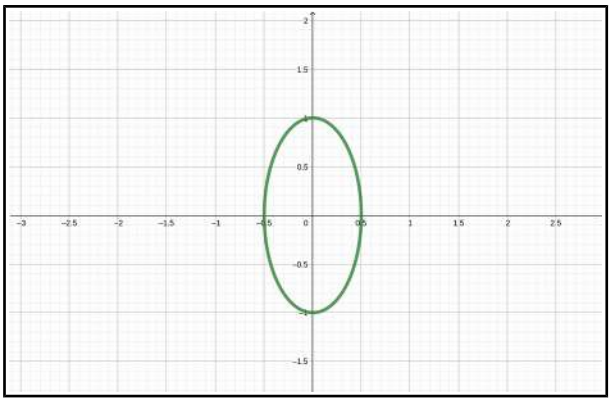


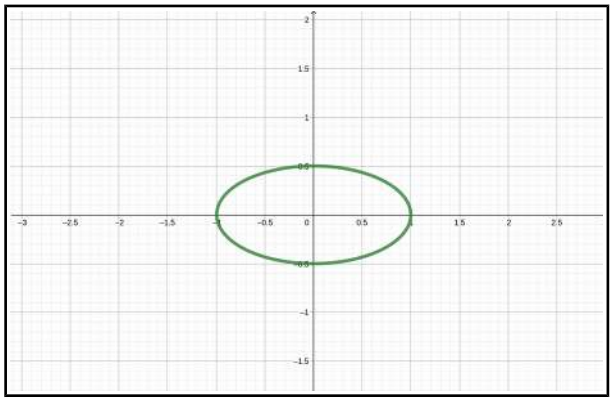
Key concept – Dilations

Let's start with a curve, for example the circle of radius 1. Its equation is_____.



If we replace ____ with _____ in the equation, we obtain a new curve with equation _____
This operation is called a **horisontal dilation** with scale factor 2. A **vertical dilation** with scale factor 2 is instead obtained by _____





The dilated curves have been stretched along the corresponding axis. If we wanted to compress them, we

_____

_____

_____

Key concept – Translations

If we want to move the curve in the Cartesian plane without deforming it, we can perform a horizontal or vertical **translation**. For example, to move the circle to the left by 1 step, we proceed by

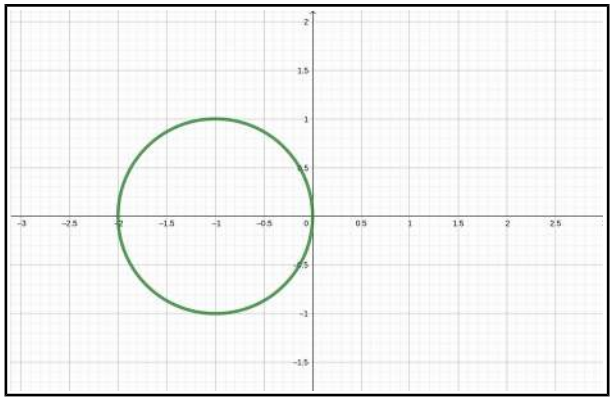
_____

_____

_____

_____

_____



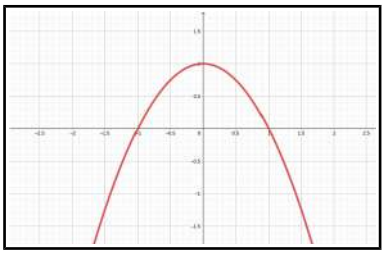
Task 2 – Plot the translated curves

Consider the parabola $y-x^2=0$. First, write the equation of the horisontal translation with step 2 to the right of the parabola. Then, translate again vertically and upwards, with step 3. Check your equations using Geogebra.

Key concept – Union

Look at the images and equations below. Discuss with the person next to you: what is the union of two curves?

$x^2+y-1=0$ $\quad$ $y-x^2+1=0$ $\quad$ $(y-x^2+1)(x^2+y-1)=0$



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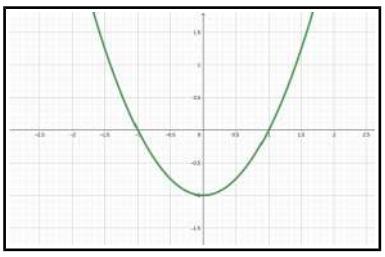

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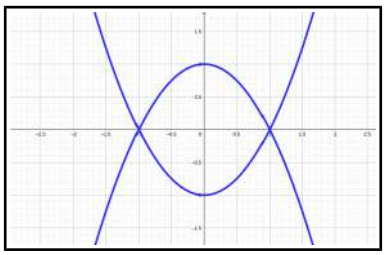
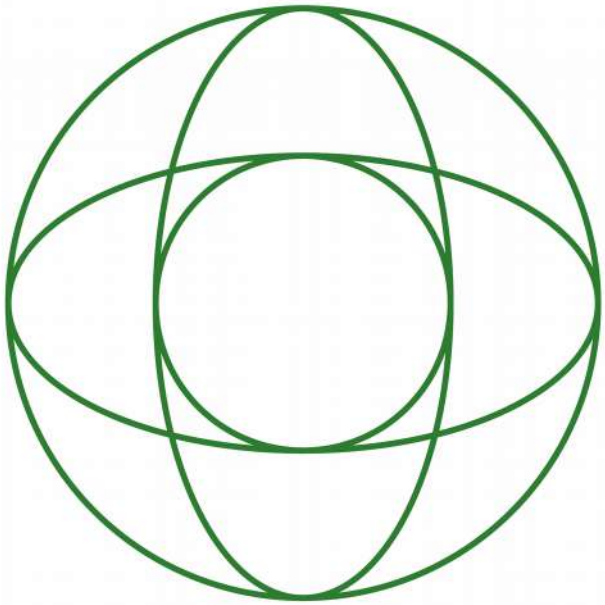


Homework

Careful: this is a bit of a challenge. You may help each other with your fellow mathematicians! Suppose you wanted to graph a curve on geogebra which is shaped like this:



How would you proceed? Try to find an equation, and explain in detail how you got to your solution or what were the steps that you took to tackle the problem.

Tip: start from the circe of radius one and make an appropriate use of dilations and unions.

Tutorial 4 – Listen to me, I've got a degree!

What is the Purpose of Tutorial 4?

- To know what the degree of a polynomial is.
- To know what the degree of a curve is.
- To know in how many points a line and a curve can meet.

Key concept – Degree

The degree of a monomial in x and y is the sum of the exponents of x and y. For example:

The degree of a polynomial is the maximal degree among its monomials. For example:

Task 1 – Find the degree

You will see around the room some banners with whole numbers. When you see a polynomial on the screen, touch the banner corresponding to its degree!

Task 2 – Group Discussion: what is a curve of degree 1?

Think, pair share. Take a minute to thik about the question, discuss with the person next to you and share with others what you think.

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Fill the following table:

General equation for non-vertical line:	
If **m=0** then...	
If **m>0** then...	
If **m<0** then...	
The meaning of **c** is...	
Equation for a vertical line:	

Task 3 – Cutting the circle

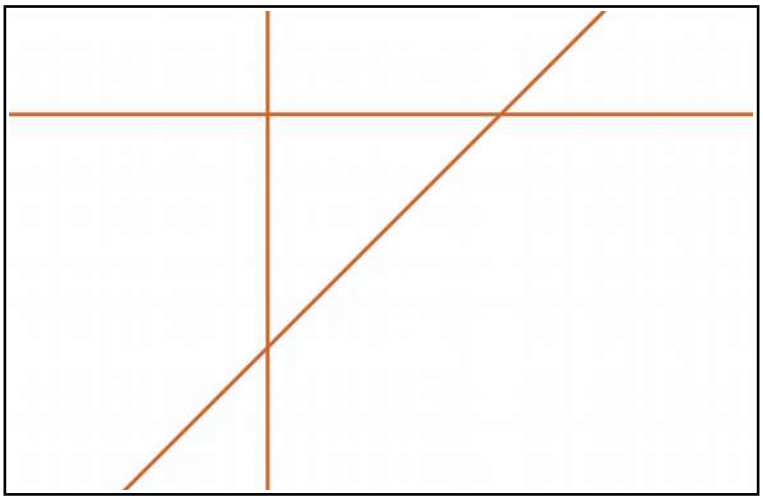
Plot on Geogebra our favourite degree 2 curve, which is the circle of radius 1 ($x^2+y^2-1=0$). Find 2 lines such that:
1) the first does not meet the circle;
2) the second meets the circle in 2 points;
3) (way harder) the third meets the circle in only 1 point.

Fact: a line and a curve of degree n meet in at most n points.

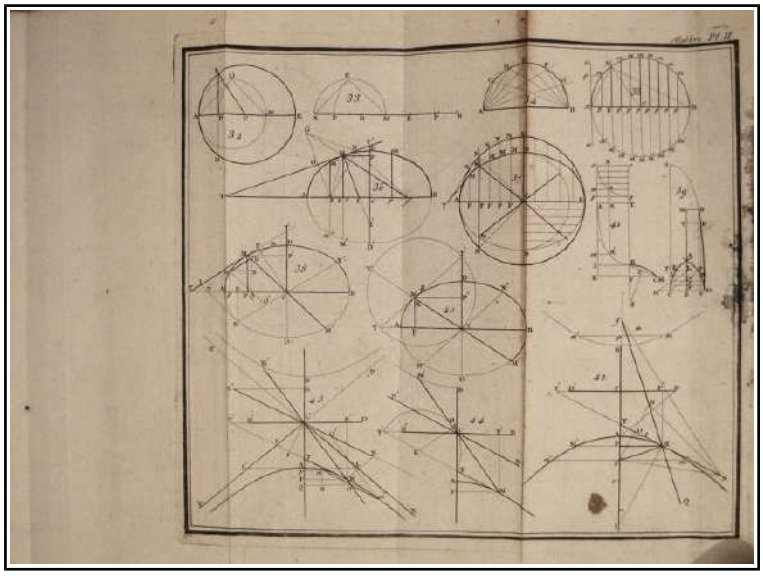
Homework

1) Plot on Geogebra the **serpentine** curve, with equation $x^2 y + y - x = 0$. What is the degree of the curve? How would you describe it? Find 4 lines meeting the curve in 0, 1, 2 and 3 points.

2) Find the equation of a curve consisting of 3 lines pairwise meeting in 3 different points, like the one below. Plot your result.



Tutorial 5 –A meeting with Bezout

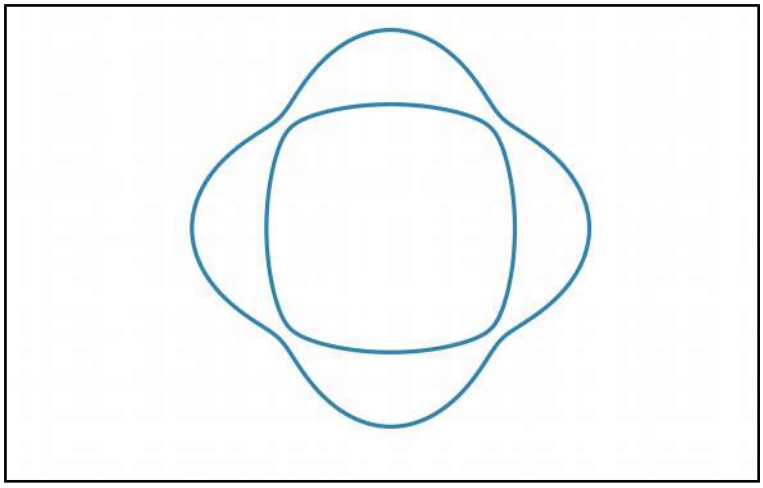


What is the Purpose of Tutorial 5?

- To know what is the content of Bezout's theorem.
- To learn how to play with animations and deformations..

Task 1 – Group Discussion

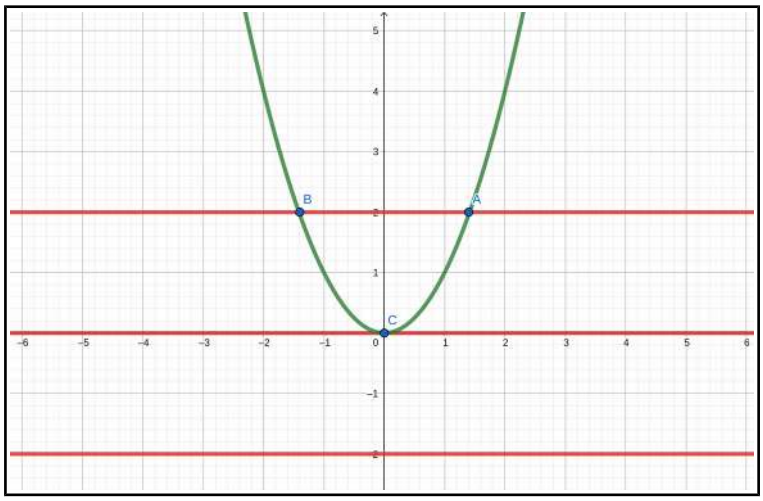
Consider the sentence: "the degree curve below must be at least 4". Do you agree? Why?



Key concept – Bezout's theorem

Suppose you have two curves of degrees n and m. In how many points do they meet?

We already know that there isn't a single answer to the question. If we start with a parabola $y-x^2=0$ (degree=2) and we ask in how many points it can meet a line (degree=1), we have 3 possible solutions: 0, 1 or 2.



And yet, we know that a line and a parabola can meet **no more** than twice.

Bezout's Theorem (1779): two plane algebraic curves of degree m and n meet in **at most** $m \cdot n$ points.

Final Task – Animations on Geogebra!

How do animations work?

Animations allow you to graph a lot of curves in one go, and to smoothly move between a curve and the other. A simple (but a bit boring) example: plug $y-x+t=0$. As you "move" t, you will see a line sliding vertically. More interesting examples are the family of elliptic curves: $y^2-x^3+x+0.2t=0$ or Klein's construction of a quartic with 4 components $(0.1x^2+9y^2-1)(9x^2+0.1y^2-1)+t=0$. Create your own animations!

Final Assignment

The final assignment consists of two parts, which will have the same weigh on your final grade. The success criteria are the same as the ones for the baseline assignment: the exposition should be clear and use appropriate language, the problems should be solved and the solutions clearly explained.

Essay: what is a plane algebraic curve?

Write a short essay (between 500 and 1000 words) in response to the above question. Your answer should include as much as possible of the following.
a) The meaning of the word **polynomial** in x and y.
b) The formal definition of a curve as set of **points satisfying a polynomial equation**.
c) A description of the meaning of the words: **bounded, singular, component, smooth and degree**.
d) A presentation of **Bezout's theorem**, including the particular case in which one of the two curves is a line.
e) Many appropriate examples of curves, together with their graphs (please use **Geogebra**).

You are **strongly encouraged** to do your own research and create your own examples. I will send you via email a list of useful websites to help you with your essay.

Problem set

Solve as many as you can of the following exercises. For each exercise, explain in detail how you found the solution. Include graphs when required.

1) Find a curve of degree 4 with 1 component and 3 singular points and plot it.

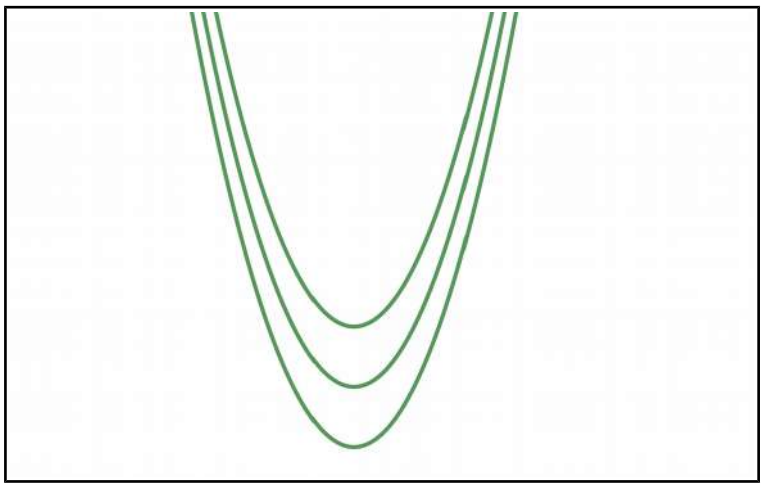
2) Find a bounded curve of degree 4 with 2 singular points and plot it.

3) Plot the curve $x^2+y^2+1=0$ with Geogebra. What happens? Can you explain why?

4) Find a bounded nonsingular curve of degree 4 with 2 components and plot it.

5) Simplify the expression $x(x-y^2)-2y(xy+y)+2y^2+(x-y)(x+y)$

6) Look at the curve below. Explain why its degree is **at least** 6, find the equation of a curve with this shape and plot it.



Tutorial 6 – Feedback tutorial

What is the Purpose of Tutorial 6?

- To receive feedback on final assignments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement for the final assignment.

Final assignment feedback

What I did well...	What I could have improved on...
•	•
•	•
•	•

My target for future work is...

Tutorial 7 – Final feedback tutorial

What is the Purpose of Tutorial 7?

- To receive final feedback on final assignments.
- To reflect on the programme including what was enjoyed and what was challenging.

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?
• • •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

Appendix 1 – Referencing correctly

When you get to university, you will need to include references in the assignments that you write, so we would like you to start getting into the habit of referencing in your Brilliant Club assignment. This is really important, because it will help you to avoid plagiarism. Plagiarism is when you take someone else's work or ideas and pass them off as your own. Whether plagiarism is deliberate or accidental, the consequences can be severe. In order to avoid losing marks in your final assignment, or even failing, you must be careful to reference your sources correctly.

What is a reference?

A reference is just a note in your assignment which says if you have referred to or been influenced by another source such as book, website or article. For example, if you use the internet to research a particular subject, and you want to include a specific piece of information from this website, you will need to reference it.

Why should I reference?

Referencing is important in your work for the following reasons:

- It gives credit to the authors of any sources you have referred to or been influenced by.
- It supports the arguments you make in your assignments.
- It demonstrates the variety of sources you have used.
- It helps to prevent you losing marks, or failing, due to plagiarism.

When should I use a reference?

You should use a reference when you:

- Quote directly from another source.
- Summarise or rephrase another piece of work.
- Include a specific statistic or fact from a source.

How do I reference?

There are a number of different ways of referencing, and these often vary depending on what subject you are studying. The most important to thing is to be consistent. This means that you need to stick to the same system throughout your whole assignment. Here is a basic system of referencing that you can use, which consists of the following two parts:

1. **A marker in your assignment**: After you have used a reference in your assignment (you have read something and included it in your work as a quote, or re-written it your own words) you should mark this is in your text with a number, e.g. [1]. The next time you use a reference you should use the next number, e.g. [2].
2. **Bibliography**: This is just a list of the references you have used in your assignment. In the bibliography, you list your references by the numbers you have used, and include as much information as you have about the reference. The list below gives what should be included for different sources.

    a. **Websites** – Author (if possible), title of the web page, website address, [date you accessed it, in square brackets].

    E.g. Dan Snow, 'How did so many soldiers survive the trenches?', http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [11 July 2014].

    b. **Books** – Author, date published, title of book (in italics), pages where the information came from.

    E.g. S. Dubner and S. Levitt, (2006) *Freakonomics*, 7-9.

    c. **Articles** – Author, 'title of the article' (with quotation marks), *where the article comes from* (newspaper, journal etc.), date of the article.

    E.g. Maev Kennedy, 'The lights to go out across the UK to mark First World War's centenary', Guardian, 10 July 2014.

Notes

Notes

Notes

A not so straight introduction to curves

Key Stage 4 Programme

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