Researchers in Schools

How to get rich solving a maths problem

Key Stage 4 Programme

Pupil Name

Coursebook Designed by **Dr Smith**

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1	03.12.2019	12:00pm-3:45pm	The University of Cambridge
2	09.01.2019	3:45pm-4:45pm	
3	16.01.2019	3:45pm-4:45pm	
4	23.01.2019	3:45pm-4:45pm	
5	30.01.2019	3:45pm-4:45pm	
6 (Feedback)			
7 (Feedback)			

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1	Baseline assignment	10.12.2019
Tutorial 2	Functions of 2 variables	16.01.2019
Tutorial 3	Limits of sequences	23.01.2019
Tutorial 4	Summing all the whole numbers	30.01.2019
Tutorial 5	Draft final assignment	

Assignment Submission – Lateness and Plagiarism

Lateness	
	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

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Course Rationale

Did you know that you can win $1,000,000 by solving a maths problem? In this course we will learn about prime numbers, infinite sums and functions of 2 variables, and relate this to arguably the most important unsolved problem of all time.

At the turn of the millennium $1,000,000 was attached to each of 7 important mathematical problems.

There are many unsolved problems in mathematics, and whilst studying this course we will develop an appreciation for the evolving nature of the subject.

In this course we will focus on one of the so called millennium problems. We will explore the Riemann hypothesis, which was first conceived in 1859, and show how an understanding of mathematical objects called L-functions could hold the key to solving the mystery of this $1,000,000 problem.

Uni Pathways Mark Scheme 2019 – problem-set assignments

	Skills	1st (70-100)	2:1 (60-69)	2:2 (50-59)	3rd (40-49)	Mark /100
Subject knowledge	Knowledge and understanding	o All content included and materials used are relevant to the general topic and to the specific question/title o Good understanding of all the relevant topics. o Technical terms are defined and used accurately throughout o Clear justification of how the material and content included is related to the specific issues that are the focus of the assignment	o Most of the content included is relevant to the general topic and to the specific question/title o Good understanding of most the relevant topics o Technical terms are used accurately but not always clearly defined. o Adequate justification of how the content included is related to the specific issues that are the focus of the assignment	o Some of the content included is relevant to the general topic and to the specific question/title o Good understanding on some of the relevant topics but occasional confusion on others. o Scientific terms are used mostly accurately with occasional confusion and often not defined. o Some justification on how the content included is related to the specific issues that are the focus of the assignment	o The content included and materials used are not applied to the question/title in a relevant manner o There is confusion in how understanding of the topics is expressed	
Critical thinking	Research and evidence	o Uses evidence/calculations to support claims/assertions/ideas, consistently clearly and convincingly o Data is effectively analysed, and appropriate assumptions/conclusions are reached o Includes rich sources of research findings, data, quotations or other sourced material as evidence for the claims/ideas	o Uses evidence/calculations to support claims/assertions/ideas, mostly clearly and convincingly o Data is analysed and the assumptions/conclusions that are reached are mostly appropriate o Includes adequate sources of research findings, data, quotations or other sourced material as evidence for the claims/ideas	o Uses evidence/calculations to support claims/assertions/ideas, at times clearly and convincingly o There is an attempt to analyse data is and draw assumptions/conclusions o Includes some sources of research findings, data, quotations or other sourced material as evidence for the claims/ ideas	o Data is presented largely without analysis	
	Developing an argument	o Argument/proof is <u>exceptionally</u> well-developed and well-justified o Uses content from the tutorials in an unfamiliar context and does so accurately and confidently.	o Argument/proof is <u>clear and well-developed,</u> and position is justified o Uses some content from the tutorials in an unfamiliar context, but not always accurately	o Argument/proof is <u>clear but not well-developed</u> o Limited use of content from the tutorials in other contexts	o Sometimes the argument/proof is not clearly established o Limited use of content from the tutorial and often inaccurately	
	Critical evaluation	o Effectively critiques the problem set and effectively establishes a thorough response to it	o Some evidence of critiquing the problem set and establishes a reasonable response to it	o Limited evidence of critiquing the problem set and establishes a response to it that has some coherence	o The problem is explored in the work but without explanation	

Written communication	Structure and presentation	○ Ideas are presented in paragraphs and arranged in a logical structure that is appropriate for the assignment ○ The introduction clearly outlines how the essay/report will deal with the issues ○ The conclusion summarises all the main points clearly and concisely ○ All calculations, formulas and methods are clearly structured, clear to follow and correct ○ Tables and graphs are effectively constructed including appropriate headings, units and scales. ○ All sources are referenced correctly in an agreed format	○ Ideas are presented in paragraphs and arranged in a structure that is mostly appropriate for the assignment ○ The introduction adequately describes how the essay/report will deal with the issues ○ The conclusion summarises most of the main points clearly ○ Calculations, formulas and methods are mostly structured, clear to follow and correct ○ Most tables and graphs are well constructed ○ Most sources are referenced correctly in an agreed format	○ Ideas are presented in paragraphs and arranged in a structure ○ The introduction mentions how the essay/report will deal with the issues ○ The conclusion summarises some of the main points clearly ○ Calculations, formulas and methods are not always structured, clear to follow and correct. ○ Some tables and graphs are well constructed but contains some errors ○ Some sources are referenced correctly in the agreed format with occasional errors	○ Ideas are presented in paragraphs but there is a lack of structure in how the work is presented ○ The work lacks an introduction that establishes the scope of the question ○ The work lacks a conclusion that summarise the main points raised ○ Work is not referenced accurately	
	Language and style	○ No spelling, grammar or punctuation errors ○ Writing style consistently clear, tone appropriate and easy to follow ○ Accurate and consistent use of technical language and vocabulary	○ Minimal spelling, grammar or punctuation errors ○ Writing style mostly clear, tone appropriate and easy to follow ○ Some attempts of using technical language and vocab alary, but not always accurate	○ Some spelling, grammar or punctuation errors ○ Writing style moderately clear, tone appropriate and easy to follow ○ Use of simple language and vocabulary effectively but struggles to use technical language	○ There are a significant number of spelling, grammar and punctuation errors ○ Use of simple language and vocabulary effectively but a lack of technical language	

Overall Mark for written element (average of the 6 marks from the criteria above)/100	
	+
Problem Set mark/100	
	÷ 2
Final mark/100	

Baseline Assignment (problem-set): Pupil Feedback Report

Name of Pupil	
Name of School	
Name of RIS teacher	
Title of Assignment	What do we know about the twin prime/Goldbach conjecture (deleting where appropriate)?

How your assignment is graded:

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-level
2:1	60-69	Performing to a good standard at A-level
2:2	50-59	Performing to an excellent standard at GCSE
3rd	40-49	Performing to a good standard at GCSE
Working towards a pass	0-39	Performing below a good standard at GCSE
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Marks

OVERALL MARK / 100		FINAL MARK / 100(including any deductions)	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the teacher should give an explanation in this section:

Mark Breakdown and Feedback

Knowledge and Understanding		Research and Evidence	
	mark		*mark*
Developing an Argument		**Critical Evaluation**	
	mark		*mark*
Structure and Presentation		**Language and Style**	
	mark		*mark*
Problem Set			

Final Assignment (problem-set): Pupil Feedback Report

Name of Pupil	
Name of School	
Name of RIS teacher	
Title of Assignment	

How your assignment is graded:

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-level
2:1	60-69	Performing to a good standard at A-level
2:2	50-59	Performing to an excellent standard at GCSE
3rd	40-49	Performing to a good standard at GCSE
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Marks

OVERALL MARK / 100		FINAL MARK / 100(including any deductions)	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the teacher should give an explanation in this section:

Mark Breakdown and Feedback

Knowledge and Understanding		Research and Evidence	
	mark		*mark*
Developing an Argument		**Critical Evaluation**	
	mark		*mark*
Structure and Presentation		**Language and Style**	
	mark		*mark*
Problem Set			

Subject Vocabulary

Word	Definition	In a sentence
Conjecture	An informed mathematical guess that has not been verified to be true.	
Divisor	A number that divides another	3 is a divisor of 6 as 6 = 2 x 3.
Factorial		
Function	A rule to assign an input to an output number.	
Limit		
Prime number	A number (greater than 1) with only 1 and itself as divisors.	
Product	The product of 2 numbers is found by multiplying them together.	The product of 2 and 3 is 6.
Quadrant	A quadrant of a graph is ¼ of the graph formed from the intersection of the axes.	
Summation		
Twin prime	A pair of primes separated by 2.	(3,5), (5,7) and (11,13) are all twin primes.

Tutorial 1 - Open problems

What is the Purpose of Tutorial 1?

By the end of this tutorial I will be able to…

- Recognise that there are many important unsolved problems in mathematics, as exemplified by the millennium prize problems
- Understand the argument that tells us the primes do not run out
- Describe the notion of a twin prime and describe the twin prime conjecture

Can you identify

1. The first 10 prime numbers
2. Any pairs of primes separated by 2 or 4
3. Any pairs of primes separated by 3 or 5?

Why is it almost always the case that pairs of primes are separated by an even number?

The millennium problems:

There are many unsolved problems in mathematics. In this course we will focus on a problem with a reward of $1 000 000 for its solution.

Activity 1: Can we run out of primes?

With a pencil you should fill in the blanks with the words provided.

If the primes run out we can _____ them as p_1, p_2, ... and p_n.

The number $p_1 \times p_2 \times \ldots p_n + 1$ is not _____ by p_1, p_2, ... or p_n as it leaves a remainder of _____ after being divided by p_1, p_2, ... or p_n.

Therefore there must be a prime that _____ be in the list: the primes _____ run out.

Words: divisible, add, multiplied, can, cannot, one, two, list

Unsolved problem 1 - the twin prime conjecture:

The twin prime conjecture says that the twin primes do not run out.

Activity 2:

What are 2, 4, 6, 8 and 10 written as the sum of 2 primes? Can any be written in more than 1 way?

Unsolved problem 2 - the Goldbach conjecture:

The Goldbach conjecture says that every even number can be written as the sum of two primes.

Baseline assignment:

Title:

What do we know about the twin prime/Goldbach conjecture (deleting where appropriate)?

Instructions:

Problem set:

Fully worked solutions should be provided to the questions.

Written report:

At least 250 words should be written on 1 of the open problems introduced in the tutorial.

Baseline assignment success criteria:

Knowledge and understanding: Include content that is relevant to the topic and title. Include key words from the glossary and demonstrate that you understand them by explaining what they mean and how they relate to the topic.

Research and interest: Include research findings from a wide range of sources. As well as using the sources provided, you should be discovering some sources yourself from the independent work that you do outside of tutorials.

Critical evaluation: Include critique and evaluation of the problem set.

Structure and preparation: Organise your ideas in paragraphs with a logical structure. Include an introduction that clearly defines what the assignment will contain. Make sure your calculations, formulas and methods are clear and structured correctly. Make sure you have referenced everything correctly.

Language and style: Make sure your spelling and grammar are accurate. Use a formal style of writing that is appropriate for an academic essay.

Problem set

1.

Written in the standard convention $2^{n_1} \times 3^{n_2} \times \ldots$, where $n_1, n_{2,\ldots}$ are whole numbers, what are the prime factorisations of

i)27

ii)32

iii)24?

2. If $f(x) = x^2$ is the rule that f squares x (so that $f(2) = 2^2 = 4$ for example), then what are

i)$f(0)$

ii) $f(3)$ and

iii) $f(-3)$?

3. If $g(x) = x^3 + 1$, then what are

i)$g(0)$

ii) $g(3)$ and

iii) $g(-3)$?

4. What do sketches of $f(x)$ and $g(x)$ look like?

5. If $\sum_{k=1}^{n} a_n$ means$a_1 + a_2 + \cdots + a_n$, then what are

i) $\sum_{k=1}^{3} k$

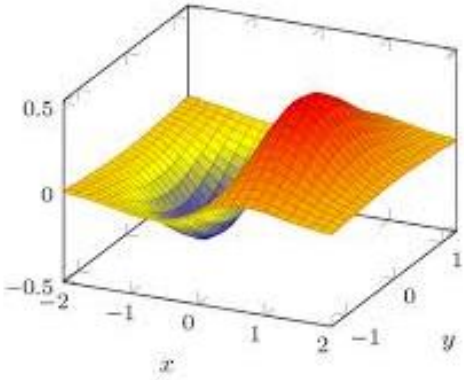
ii) $\sum_{k=1}^{n} 1$ and

iii) $\sum_{k=1}^{n} k$?

6. What is $\sum_{k=1}^{\infty} \frac{1}{2^k}$?

Can any meaning be given to $\sum_{k=1}^{\infty} \frac{1}{k^2}$, $\sum_{k=1}^{\infty} \frac{1}{k}$ or $\sum_{k=1}^{\infty} k$?

Tutorial 2 – Functions as landscapes



By the end of this tutorial I will be able to…

- Factorise whole numbers into their standard prime factorisation
- Evaluate and plot functions of 1 and 2 variables

Can you identify

1. All factors of 24
2. All prime factors of 24
3. The prime factorisation of 24?

Function machines:

A function machine takes an input and performs operations on this to give an output.

Can you give an example of a function machine?

In mathematical notation, we can write the input as x and the function as f (we don't have to use these letters but seem sensible. The final output will then be written as f(x).

Activity 1:

Using the notation introduced you should give an example of a function (you may use the function machine you thought of earlier.

Think of a number:

Think of any number (but not too large that you'll struggle to do some mental arithmetic with it). Now follow the steps:

1. Double it
2. Add 6
3. Double it again
4. Divide it by 4
5. Take away the number you started with

You should all have got the same number (3) at the end!

Activity 2:

Identify the function at each step of the process and draw the graph of 1 of them.

Functions of 2 variables:

The input of a function of 2 variables is an ordered pair, (x,y). As an example, we could have $f(x,y) = 2x + y^2$. Then (0,0) goes to 0, as $2 \times 0 + 0^2 = 0$ and (1,1) goes to 3, as $2 \times 1 + 1^2 = 2 + 1 = 3$.

Order matters!

What are f(0,1) and f(1,0)?

Activity 3:

Another example of a function of 2 variables is 1 when x is positive and 0 if x is negative.

Can you define the function precisely and sketch it on a graph?

Homework title: Functions of 2 variables

Homework instructions: An example of a function of 2 variables should be constructed and plotted below

Homework success criteria:

Knowledge and understanding: Include content that is relevant to the topic and title. Include key words from the glossary and demonstrate that you understand them by explaining what they mean and how they relate to the topic.

Research and interest: Include research findings from a wide range of sources. As well as using the sources provided, you should be discovering some sources yourself from the independent work that you do outside of tutorials.

Tutorial 3 – Infinite series

By the end of this tutorial I will be able to…

- Recognise a geometric sequence and the formula for its summation
- Calculate the limits of simple sequences and understand how the formula for the infinite summation of a geometric sequence can be used

Sequences and series

An arithmetic sequence is one with a fixed difference between consecutive terms, and a geometric sequence is one with a fixed ratio between consecutive terms.

Can you identify which of the following are arithmetic or geometric sequences
1. 1, 2, 3, ...
2. 1, ½, ¼, ...
3. 1, 1, 1, …?

Summation of sequences:

There is a formula to sum arithmetic and geometric sequences. For an arithmetic sequence the sum is given by adding the first and last terms and then multiplying this by half the number of terms.

Activity 1:

You should work out the sum for the first starter question where the last term is 10 and 100.

Activity 2:

What is the sum of the terms in the second starter question? By adding enough terms can you exceed 2?

What is the conclusion?

The formula for the sum of a geometric sequence is slightly more complicated than the one for the arithmetic case:

$$s = \frac{a(r^{n-1} - 1)}{r^{n-1}}.$$

Activity 3:

Check:

The formula provided for the geometric case can now be used to check the answers found earlier by adding the terms one at a time.

We can see that the formula simplifies if the number of terms is taken to infinity but the ratio must not have a size greater than 1.

Homework title: Limits of sequences

Instructions:

Full working should be given to determine the limit as n tends to infinity of

1. $\frac{2n}{n+1}$
2. $\frac{6n^2+4n^2}{2n^2+1}$.

The formula for the sum of the geometric sequence should be researched and the steps in its proof written out.

Tutorial 4 – A $1,000,000 problem

What is the Purpose of Tutorial 4? By the end of this tutorial I will be able to…

- Recognise the Riemann zeta function as a function of 2 variables that describes a rich landscape
- Describe some key known and conjectured properties of the Riemann landscape

The factorial function of a whole number; n, is calculated by finding the product of all whole numbers up to n. The factorial of n, or n factorial, is written n!.

Can you calculate the following without a calculator

1. 4!
2. 5!
3. 10!/8!?

Analytic continuation:

In order for the factorial function to be smooth when taken over all numbers, there is in fact a unique way to calculate it. To do so introduces a function called the gamma function, which requires some A level maths to understand.

The Riemann zeta function is a function of 2 variables expressed as an infinite sum. Like the factorial function it isn't originally defined for all variables but can be extended.

Activity 1:

Key areas of input for Riemann zeta function should be labelled on a graph. These include the trivial zeros, the critical strip and the critical line.

Homework title: Summing all the whole numbers

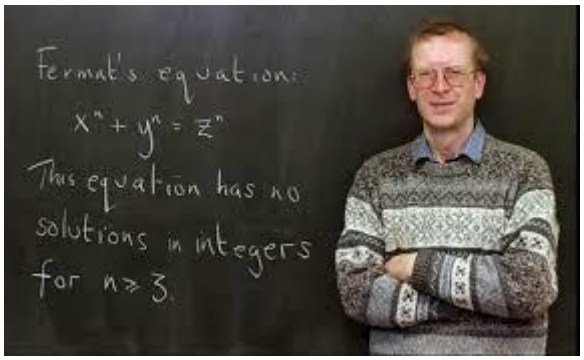
Instructions: With reference to the analytic continuation of the Riemann zeta function at $x = -1$, you should explain in a referenced piece of writing, how it can be sensible to attribute a negative number with the sum of all the positive whole numbers.

Homework success criteria:

Knowledge and understanding: Include content that is relevant to the topic and title. Include key words from the glossary and demonstrate that you understand them by explaining what they mean and how they relate to the topic.

Research and interest: Include research findings from a wide range of sources. As well as using the sources provided, you should be discovering some sources yourself from the independent work that you do outside of tutorials.

Tutorial 5 – L-functions: the key



By the end of this tutorial I will be able to…

- Appreciate the use of Dirichlet L-functions in understanding primes in arithmetic progressions
- Identify another example of an L-function

Can you identify

1. The first 10 primes in the arithmetic sequence: 2, 5, 8, 11, 14, ...
2. The first 10 primes in the arithmetic sequence: 3, 7, 11, 15, 19, …?

Dirichlet L-functions

Dirichlet L-functions share many properties with the Riemann zeta function and were used to prove that the primes don't run out even if we restrict ourselves to arithmetic sequences.

Activity 1:

The internet should be used to find 3 properties in common with the Riemann zeta function. A good starting point is http://www.lmfdb.org/.

Activity 2:

Go to http://mathworld.wolfram.com/RiemannZetaFunction.html and use the graphing calculator to produce 3 Riemann landscapes, zooming in to different extents. Notes and observations should be made in the space below.

Activity 3:

Another example of an L-function should be found together with 3 similar properties to those found for the previous examples.

Final assignment:

Title: The landscape of L-functions

Instructions:

Problem set:

Fully worked solutions should be provided to the questions.

Written report:

At least 1 000 words should be written in a referenced report on L-functions.

Success criteria:

Knowledge and understanding: Include content that is relevant to the topic and title. Include key words from the glossary and demonstrate that you understand them by explaining what they mean and how they relate to the topic.

Research and interest: Include research findings from a wide range of sources. As well as using the sources provided, you should be discovering some sources yourself from the independent work that you do outside of tutorials.

Critical evaluation: Include critique and evaluation of the problem set.

Structure and preparation: Organise your ideas in paragraphs with a logical structure. Include an introduction that clearly defines what the assignment will contain. Make sure your calculations,
formulas and methods are clear and structured correctly. Make sure you have referenced everything correctly.

Language and style: Make sure your spelling and grammar are accurate. Use a formal style of writing that is appropriate for an academic essay.

Problem set

1. Written in the standard convention $2^{n_1} \times 3^{n_2} \times \ldots$, where $n_1, n_{2,\ldots}$ are whole numbers, what are the prime factorisations of

i)16

ii)60

iii)42?

2. If $f(x) = x^2 - 1$ and $g(x) = x^3$, then what are

i)$f(0)$

ii) $f(3)$ and

iii) $f(g(-1)$?

3. What do sketches of $f(x)$ and $g(x)$ look like?

4. What are

i) $\sum_{k=1}^{n} x^k$

ii) $\sum_{k=0}^{\infty} x^k$

iii) $\sum_{k=1}^{\infty} kx^k$

iv) $\sum_{k=1}^{\infty} \frac{1}{k^4}$ and

v) $\sum_{k=1}^{\infty} \frac{1}{k^6}$?

vi) What do we know about the infinite sums of reciprocals of the odd powers? Please aim to write at least 2 sentences.

Tutorial 6 – Feedback tutorial

What is the Purpose of Tutorial 6?

- To receive feedback on your final assignment
- To respond to the feedback from your Uni Pathways teacher
- To write targets for improvement on your final assignment

Final assignment feedback from your Uni Pathways Teacher

(Remember to look at the mark scheme to help you understand what you have done well so far, and how you can do even better in your final assignment)

Here are three things that my Uni Pathways Teacher thought I did well in my draft assignment
•   •   •

Here are three things that my Uni Pathways Teacher thinks that I could do to get a higher mark in my final assignment
•   •   •

Tasks from my Uni Pathways Teacher to do during the feedback tutorial to help me improve
•   •   •
My response:

Actions I will take to improve my final assignment after this tutorial…
•   •   •

Hand in date for my final assignment:	

Tutorial 7 – Final tutorial

What is the Purpose of Tutorial 7?

- To receive feedback and a grade on your final assignment.
- To reflect on the programme including what you enjoyed and what was challenging.
- To ask any questions you may have about university.

Final assignment feedback from my Uni Pathways Teacher

Final mark:	University style grade:
Feedback: Here are three things that my Uni Pathways teacher thought I did well in my final assignment • • •	

Here are three things that my Uni Pathways teacher thinks I should remember for when I am doing this kind of study in the future
• • •

University

What questions do you still have about University after taking part in Uni Pathways?
• • •

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?
• • •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

Appendix 1 – Referencing correctly

When you get to university, you will need to include references in the assignments that you write, so we would like you to start getting into the habit of referencing in your Brilliant Club assignment. This is really important, because it will help you to avoid plagiarism. Plagiarism is when you take someone else's work or ideas and pass them off as your own. Whether plagiarism is deliberate or accidental, the consequences can be severe. In order to avoid losing marks in your final assignment, or even failing, you must be careful to reference your sources correctly.

What is a reference?

A reference is just a note in your assignment which says if you have referred to or been influenced by another source such as book, website or article. For example, if you use the internet to research a particular subject, and you want to include a specific piece of information from this website, you will need to reference it.

Why should I reference?

Referencing is important in your work for the following reasons:

- It gives credit to the authors of any sources you have referred to or been influenced by.
- It supports the arguments you make in your assignments.
- It demonstrates the variety of sources you have used.
- It helps to prevent you losing marks, or failing, due to plagiarism.

When should I use a reference?

You should use a reference when you:

- Quote directly from another source.
- Summarise or rephrase another piece of work.
- Include a specific statistic or fact from a source.

How do I reference?

There are a number of different ways of referencing, and these often vary depending on what subject you are studying. The most important to thing is to be consistent. This means that you need to stick to the same system throughout your whole assignment. Here is a basic system of referencing that you can use, which consists of the following two parts:

1. **A marker in your assignment**: After you have used a reference in your assignment (you have read something and included it in your work as a quote, or re-written it your own words) you should mark this is in your text with a number, e.g. [1]. The next time you use a reference you should use the next number, e.g. [2].
2. **Bibliography**: This is just a list of the references you have used in your assignment. In the bibliography, you list your references by the numbers you have used, and include as much information as you have about the reference. The list below gives what should be included for different sources.

    a. **Websites** – Author (if possible), title of the web page, website address, [date you accessed it, in square brackets].

    E.g. Dan Snow, 'How did so many soldiers survive the trenches?', http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [11 July 2014].

    b. **Books** – Author, date published, title of book (in italics), pages where the information came from.

    E.g. S. Dubner and S. Levitt, (2006) *Freakonomics*, 7-9.

    c. **Articles** – Author, 'title of the article' (with quotation marks), *where the article comes from* (newspaper, journal etc.), date of the article.

    E.g. Maev Kennedy, 'The lights to go out across the UK to mark First World War's centenary', Guardian, 10 July 2014.

Appendix 2 – Article on the Riemann hypothesis

(https://www.ams.org/notices/200303/fea-conrey-web.pdf)

The Riemann Hypothesis

J. Brian Conrey

Hilbert, in his 1900 address to the Paris International Congress of Mathematicians, listed the Riemann Hypothesis as one of his 23 problems for mathematicians of the twentieth century to work on. Now we find it is up to twenty-first century mathematicians! The Riemann Hypothesis (RH) has been around for more than 140 years, and yet now is arguably the most exciting time in its history to be working on RH. Recent years have seen an explosion of research stemming from the confluence of several areas of mathematics and physics.

In the past six years the American Institute of Mathematics (AIM) has sponsored three workshops whose focus has been RH. The first (RHI) was in Seattle in August 1996 at the University of Washington. The second (RHII) was in Vienna in October 1998 at the Erwin Schrödinger Institute, and the third (RHIII) was in New York in May 2002 at the Courant Institute of Mathematical Sciences. The intent of these workshops was to stimulate thinking and discussion about one of the most challenging problems of mathematics and to consider many different approaches. Are we any closer to solving the Riemann Hypothesis after these efforts? Possibly. Have we learned anything about the zeta-function as a result of these workshops? Definitely. Several of the participants from the workshops are collaborating on the website (http://www.aimath.org/WWN/rh/) which provides an overview of the subject.

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Here I hope to outline some of the approaches to RH and to convey some of the excitement of working in this area at the present moment. To begin, let us examine the Riemann Hypothesis itself. In 1859 in the seminal paper "Ueber die Anzahl der Primzahlen unter eine gegebener Grösse", G. B. F. Riemann outlined the basic analytic properties of the zeta-function

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The series converges in the half-plane where the real part of s is larger than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole plane apart from a simple pole at $s = 1$. Moreover, he proved that $\zeta(s)$ satisfies an amazing *functional equation*, which in its symmetric form is given by

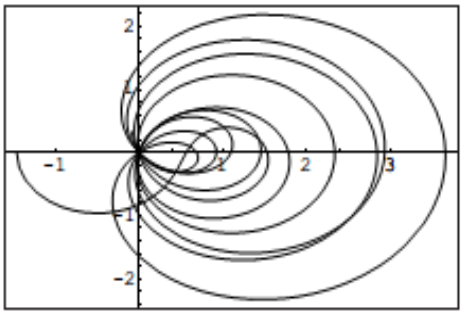


Figure 1. $\zeta(\frac{1}{2} + it)$ **for** $0 < t < 50$.

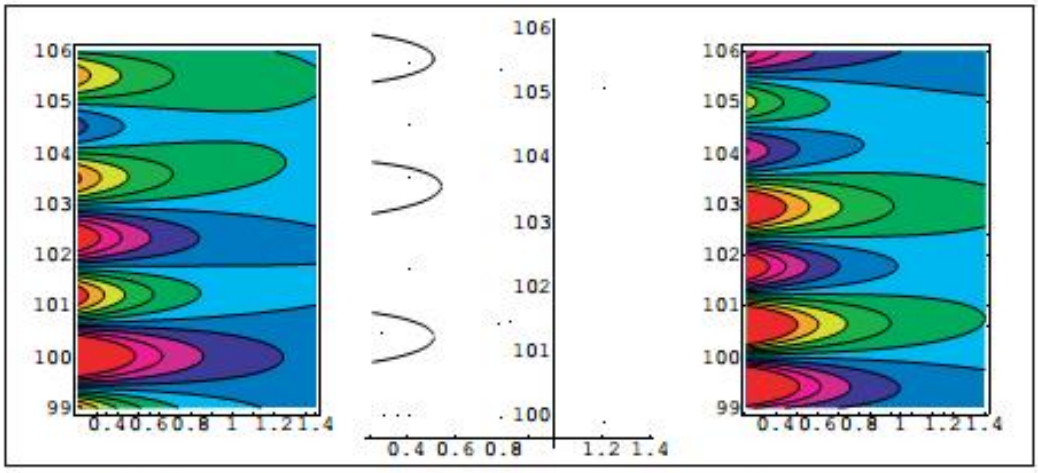


Figure 2. Contour plot of $\Re\zeta(s)$, the curves $\Re\zeta(s) = 0$ (solid) and $\Im\zeta(s) = 0$ (dotted), contour plot of $\Im\zeta(s)$.

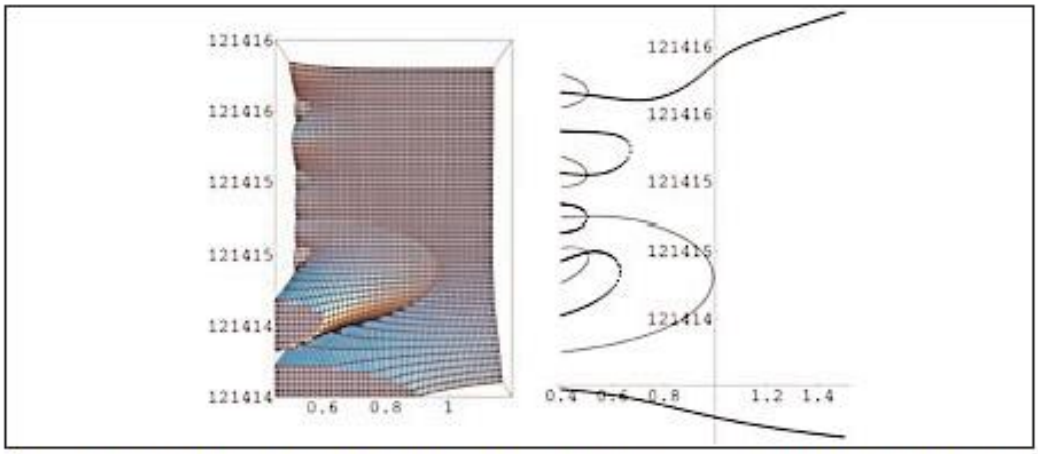


Figure 3. 3-D plot of $|\Re\zeta(s)|$, and the curves $\Re\zeta(s) = 0$ (solid) and $\Im\zeta(s) = 0$ (dotted). This may be the first place in the critical strip where the curves $\Re\zeta(s) = 0$ loop around each other.

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s),$$

where $\Gamma(s)$ is the usual Gamma-function.

The zeta-function had been studied previously by Euler and others, but only as a function of a real variable. In particular, Euler noticed that

$$\begin{aligned}\zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \dots\right) \\ &\quad \times \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right)\left(1 + \frac{1}{5^s} + \dots\right)\dots \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},\end{aligned}$$

where the infinite product (called the *Euler product*) is over all the prime numbers. The product converges when the real part of s is greater than 1. It is an analytic version of the fundamental theorem of arithmetic, which states that every integer can be factored into primes in a unique way. Euler used this product to prove that the sum of the reciprocals of the primes diverges. The Euler product suggests Riemann's interest in the zeta-function: he was trying to prove a conjecture made by Legendre and, in a more precise form, by Gauss:

$$\pi(x) := \#\{\text{primes less than } x\} \sim \int_2^x \frac{dt}{\log t}.$$

Riemann made great progress toward proving Gauss's conjecture. He realized that the distribution of the prime numbers depends on the distribution of the complex zeros of the zeta-function. The Euler product implies that there are no zeros of $\zeta(s)$ with real part greater than 1; the functional equation implies that there are no zeros with real part less than 0, apart from the *trivial zeros* at

$s = -2, -4, -6, \ldots$. Thus, all of the complex zeros are in the *critical strip* $0 \le \Re s \le 1$. Riemann gave an explicit formula for $\pi(x)$ in terms of the complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. A simpler variant of his formula is

$$\psi(x) := \sum_{n \le x} \Lambda(n)$$
$$= x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \tfrac{1}{2}\log(1 - x^{-2}),$$

valid for x not a prime power, where the von Mangoldt function $\Lambda(n) = \log p$ if $n = p^k$ for some k and $\Lambda(n) = 0$ otherwise. Note that the sum is not absolutely convergent; if it were, then $\sum_{n \le x} \Lambda(n)$ would have to be a continuous function of x, which it clearly is not. Consequently, there must be infinitely many zeros ρ. The sum over ρ is with multiplicity and is to be interpreted as $\lim_{T\to\infty} \sum_{|\rho|<T}$. Note also that $|x^{\rho}| = x^{\beta}$; thus it was necessary to show that $\beta < 1$ in order to conclude that $\sum_{n \le x} \Lambda(n) \sim x$, which is a restatement of Gauss's conjecture.

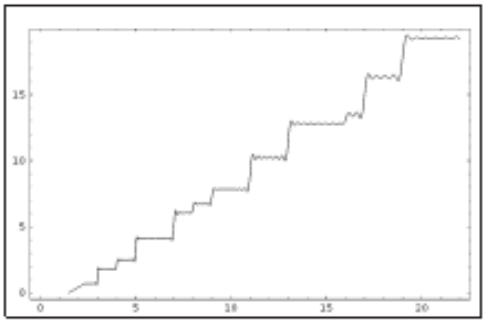


Figure 4. Explicit formula for $\psi(x)$ using the first 100 pairs of zeros.

The functional equation shows that the complex zeros are symmetric with respect to the line $\Re s = \frac{1}{2}$. Riemann calculated the first few complex zeros $\frac{1}{2} + i14.134\ldots, \frac{1}{2} + i21.022\ldots$ and proved that the number $N(T)$ of zeros with imaginary parts between 0 and T is

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(1/T),$$

where $S(T) = \frac{1}{\pi}\arg\zeta(1/2 + iT)$ is computed by continuous variation starting from $\arg\zeta(2) = 0$ and proceeding along straight lines, first up to $2 + iT$ and then to $1/2 + iT$. Riemann also proved that $S(T) = O(\log T)$. Note for future reference that at a height T the average gap between zero heights is $\sim 2\pi/\log T$. Riemann suggested that the number $N_0(T)$ of zeros of $\zeta(1/2 + it)$ with $0 < t \le T$ seemed to be about

$$\frac{T}{2\pi}\log\frac{T}{2\pi e}$$

and then made his conjecture that all of the zeros of $\zeta(s)$ in fact lie on the $1/2$-line; this is the Riemann Hypothesis.

Riemann's effort came close to proving Gauss's conjecture. The final step was left to Hadamard and de la Vallée Poussin, who proved independently in 1896 that $\zeta(s)$ does not vanish when the real part of s is equal to 1 and from that fact deduced Gauss's conjecture, now called the Prime Number Theorem.

Initial Ideas

It is not difficult to show that RH is equivalent to the assertion that for every $\epsilon > 0$,

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{1/2+\epsilon}).$$

However, it is difficult to see another way to approach $\pi(x)$ and so get information about the zeros.

Another easy equivalent to RH is the assertion that $M(x) = O(x^{1/2+\epsilon})$ for every $\epsilon > 0$, where

$$M(x) = \sum_{n \le x} \mu(n)$$

and $\mu(n)$ is the Möbius function whose definition can be inferred from its generating Dirichlet series $1/\zeta$:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

Thus, if $p_1, \ldots, p_k$ are distinct primes, then $\mu(p_1 \ldots p_k) = (-1)^k$; also $\mu(n) = 0$ if $p^2 \mid n$ for some prime p. This series converges absolutely when $\Re s > 1$. If the estimate $M(x) = O(x^{1/2+\epsilon})$ holds for every $\epsilon > 0$, then it follows by partial summation that the series converges for every s with real part greater than $1/2$; in particular, there can be no zeros of $\zeta(s)$ in this open half-plane, because zeros of $\zeta(s)$ are poles of $1/\zeta(s)$. The converse, that RH implies this estimate for $M(x)$, is also not difficult to show.

Instead of analyzing $\pi(x)$ directly, it might seem easier to work with $M(x)$ and prove the above estimate, perhaps by some kind of combinatorial reasoning. In fact, Stieltjes let it be known that he had such a proof. Hadamard, in his famous 1896 proof of the Prime Number Theorem, refers to Stieltjes's claim and somewhat apologetically offers his much weaker theorem that $\zeta(s)$ does not vanish on the 1-line in the hope that the simplicity of his proof will be useful. Stieltjes never published his proof.

Mertens made the stronger conjecture that

$$|M(x)| \le \sqrt{x};$$

clearly this implies RH. However, Mertens's conjecture was disproved by Odlyzko and te Riele in 1985. The estimate $M(x) = O(\sqrt{x})$ is also likely to

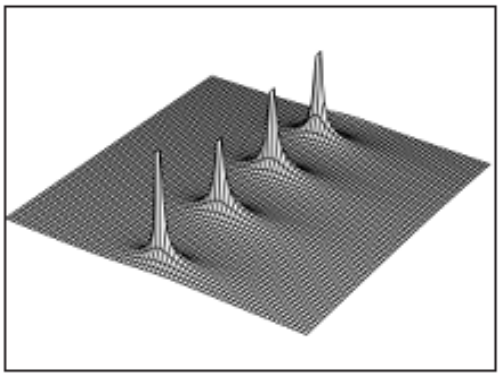


Figure 5. $1/|\zeta(x+iy)|$ **for** $0 < x < 1$ **and** $16502.4 < y < 16505$.

be false, but a proof of its falsity has not yet been found.

Subsequent Efforts

In England in the early 1900s the difficulty of the question was not yet appreciated. Barnes assigned RH to Littlewood as a thesis problem. Littlewood independently discovered some of the developments that had already occurred on the continent. Hardy, Littlewood, Ingham, and other British mathematicians were responsible for many of the results on the zeta-function in the first quarter of the century. Hardy and Littlewood gave the first proof that infinitely many of the zeros are on the 1/2-line. They found what they called the *approximate functional equation* for $\zeta(s)$. Later, Siegel uncovered a very precise version of this formula while studying Riemann's notes in the Göttingen library; the formula is now called the Riemann-Siegel formula and gives the starting point for all large-scale calculations of $\zeta(s)$. Hardy and Littlewood gave an asymptotic evaluation of the second moment of $\zeta(\frac{1}{2}+it)$; Ingham proved the asymptotics for the fourth moment.

Much effort has also been expended on the unproved Lindelöf hypothesis, which is a consequence of RH. The Lindelöf hypothesis asserts that for every $\epsilon > 0$,

$$\zeta(1/2+it) = O(t^{\epsilon}) \qquad \text{as } t \to \infty.$$

Hardy and Littlewood proved that $\zeta(1/2+it) = O(t^{1/4+\epsilon})$. This bound is now called the "convexity bound", since it follows from the functional equation together with general principles of complex analysis (the maximum modulus principle in the form of the Phragmén-Lindelöf theorem). Weyl improved the bound to $t^{1/6+\epsilon}$ with his new ideas for estimating special trigonometrical sums, now called Weyl sums.

Hardy grew to love the problem. He and Littlewood wrote at least ten papers on the zeta-function. Hardy once included proving RH on a list of New Year's goals he set for himself. Hardy even used RH as a defense: he once sent a postcard to his colleague Harald Bohr prior to crossing the English Channel one stormy night, claiming that he had solved RH. Even though Hardy was an atheist, he was relatively certain that God, if he did exist, would not allow the ferry to sink under circumstances so favorable to Hardy!

Hilbert seems to have had somewhat contradictory views about the difficulty of RH. On one occasion he compared three unsolved problems: the transcendence of $2^{\sqrt{2}}$, Fermat's Last Theorem, and the Riemann Hypothesis. In his view, RH would likely be solved in a few years, Fermat's Last Theorem possibly in his lifetime, and the transcendence question possibly never. Amazingly, the transcendence question was resolved a few years later by Gelfond and Schneider, and, of course, Andrew Wiles recently proved Fermat's Last Theorem. Another time Hilbert remarked that if he were to awake after a sleep of five hundred years, the first question he would ask was whether RH was solved.

Near the end of his career, Hans Rademacher, best known for his exact formula for the number of partitions of an integer, thought he had disproved RH. Siegel had checked the work, which was based on the deduction that a certain function would absurdly have an analytic continuation if RH were true. The mathematics community tried to get *Time* magazine interested in the story. It transpired that *Time* became interested and published an article only after it was discovered that Rademacher's proof was incorrect.

Evidence for RH

Here are some reasons to believe RH.

- Billions of zeros cannot be wrong. Recent work by van de Lune has shown that the first 10 billion zeros are on the line. Also, there is a distributed computing project organized by Sebastian Wedeniwski—a screen-saver type of program—that many people subscribe to, which claims to have verified that the first 100 billion zeros are on the line. Andrew Odlyzko has calculated millions of zeros near zeros number 10^{20}, 10^{21}, and 10^{22} (available on his website).
- Almost all of the zeros are very near the 1/2-line. In fact, it has been proved that more than 99 percent of zeros $\rho = \beta + i\gamma$ satisfy $|\beta - \frac{1}{2}| \le 8/\log|\gamma|$.
- Many zeros can be proved to be on the line. Selberg got a positive proportion, and N. Levinson showed at least 1/3; that proportion has been improved to 40 percent. Also, RH implies that all zeros of all derivatives of $\xi(s)$ are on the 1/2-line. It has been shown that more than 99 percent of the zeros of the third derivative $\xi'''(s)$ are on the 1/2-line. Near the end of his life, Levinson thought he had a method that allowed for a converse to Rolle's theorem in

this situation, implying that if $\xi'(s)$ has at least a certain proportion of zeros on the line, then so does ξ and similarly for ξ'' to ξ' and so on. However, no one has been able to make this argument work.

- Probabilistic arguments. For almost all random sequences of -1's and $+1$'s, the associated summatory function up to x is bounded by $x^{1/2+\epsilon}$. The Möbius sequence appears to be fairly random.
- Symmetry of the primes. RH tells us that the primes are distributed in as nice a way as possible. If RH were false, there would be some strange irregularities in the distribution of primes; the first zero off the line would be a very important mathematical constant. It seems unlikely that nature is that perverse!

Various Approaches

There is an often-told story that Hilbert and Pólya independently suggested that the way to prove RH was to interpret the zeros spectrally, that is, to find a naturally occurring Hermitian operator whose eigenvalues are the nontrivial zeros of $\zeta(1/2+it)$. Then RH would follow, since Hermitian operators have real eigenvalues. This idea has been one of the main approaches that has been tried repeatedly.

We describe an assortment of other interesting approaches to RH.

Pólya's Analysis

Pólya investigated a chain of ideas that began with Riemann: namely, studying the Fourier transform of $\Xi(t) := \xi(\frac{1}{2}+it)$, which as a consequence of the functional equation is real for real t and an even function of t. RH is the assertion that all zeros of Ξ are real. The Fourier transform can be computed explicitly:

$$\begin{aligned}\Phi(t) &:= \int_{-\infty}^{\infty} \Xi(u)e^{itu}\,du \\ &= \sum_{n=1}^{\infty}(2n^4\pi^2\exp(9t/2) - 3n^2\pi\exp(5t/2)) \\ &\qquad \times \exp\left(-\pi n^2 e^{2t}\right).\end{aligned}$$

It can be shown that Φ and Φ' are positive for positive t. One idea is to systematically study classes of reasonable functions whose Fourier transforms have all real zeros and then try to prove that $\Xi(t)$ is in the class. A sample theorem in this direction is due to de Bruijn:

Let $f(t)$ be an even nonconstant entire function of t such that $f(t) \geq 0$ for real t and $f'(t) = \exp(\gamma t^2)g(t)$, where $\gamma \geq 0$ and $g(t)$ is an entire function of genus ≤ 1 with purely imaginary zeros only. Then $\Psi(z) = \int_{-\infty}^{\infty} \exp\{-f(t)\}e^{izt}\,dt$ has real zeros only.

In particular, all the zeros of the Fourier transform of a first approximation (see Titchmarsh for details)

$$\begin{aligned}\phi(t) = &\left(2\pi\cosh\frac{9t}{2} - 3\cosh\frac{5t}{2}\right) \\ &\times \exp(-2\pi\cosh 2t)\end{aligned}$$

to $\Phi(t)$ are real. These ideas have been further explored by de Bruijn, Newman, D. Hejhal, and others. Hejhal (1990) has shown that almost all of the zeros of the Fourier transform of any partial sum of $\Phi(t)$ are real.

Probabilistic Models

Researchers working in probability are intrigued by the fact that the ξ-function arises as an expectation in a moment of a Brownian bridge:

$$2\xi(s) = E(Y^s)$$

where

$$Y := \sqrt{\frac{2}{\pi}}\left(\max_{t\in[0,1]} b_t - \min_{t\in[0,1]} b_t\right)$$

with $b_t = \beta_t - t\beta_1$ where β_t is standard Brownian motion. See a paper of Biane, Pitman, and Yor (*Bull. Amer. Math. Soc.* (N.S.) **38** (2001), 435–65).

Functional Analysis: The Nyman-Beurling Approach

This approach begins with the following theorem of Nyman, a student of Beurling.

RH holds if and only if

$$\operatorname{span}_{L^2(0,1)}\{\eta_\alpha, 0 < \alpha < 1\} = L^2(0,1)$$

where

$$\eta_\alpha(t) = \{\alpha/t\} - \alpha\{1/t\}$$

and $\{x\} = x - [x]$ is the fractional part of x.

This has been extended by Baez-Duarte, who showed that one may restrict attention to integral values of $1/\alpha$. Balazard and Saias have rephrased this in a nice way:

RH holds if and only if

$$\inf_A \int_{-\infty}^{\infty} \left|1 - A(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)\right|^2 \frac{dt}{\frac{1}{4}+t^2} = 0,$$

where the infimum is over all Dirichlet polynomials A.

Let d_N be the infimum over all Dirichlet polynomials

$$A(s) = \sum_{n=1}^{N} a_n n^{-s}$$

of length N. They conjecture that $d_N \sim C/\log N$, where $C = \sum_\rho 1/|\rho|^2$. Burnol has proved that

$$d_n \geq \frac{1}{\log N}\sum_{\rho \text{ on the line}} \frac{m_\rho^2}{|\rho|^2},$$

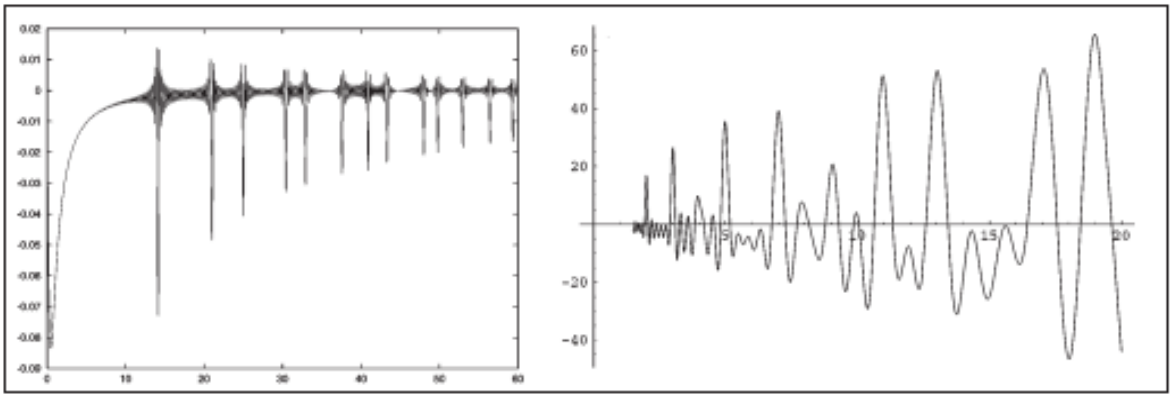


Figure 6. Duality: The Fourier transform of the error term in the Prime Number Theorem (note the spikes at ordinates of zeros) and the sum over zeros $-\sum x^\rho$ with $|\rho| < 100$ (note the peaks at primes and prime powers).

where m_ρ is the multiplicity of the zero ρ. If RH holds and all the zeros are simple, then clearly these two bounds are the same.

Weil's Explicit Formula and Positivity Criterion

André Weil proved the following formula, which is a generalization of Riemann's formula mentioned above and which specifically illustrates the dependence between primes and zeros. Suppose h is an even function that is holomorphic in the strip $|\Im t| \le 1/2 + \delta$ and that satisfies $h(t) = O((1+|t|)^{-2-\delta})$ for some $\delta > 0$, and let

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iur}\,dr.$$

Then we have the following duality between primes and zeros of ζ:

$$\begin{aligned}\sum_\gamma h(\gamma) = {} & 2h(\tfrac{i}{2}) - g(0)\log\pi \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(\tfrac{1}{4}+\tfrac{1}{2}ir)\,dr \\ & - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n).\end{aligned}$$

In this formula, a zero is written as $\rho = 1/2 + i\gamma$ where $\gamma \in \mathbb{C}$; of course RH is the assertion that every γ is real. Using this duality Weil gave a criterion for RH:

RH holds if and only if

$$\sum_\gamma h(\gamma) > 0$$

for every (admissible) function h of the form $h(r) = h_0(r)\overline{h_0(\overline{r})}$.

Xian-Jin Li has given a very nice criterion which, in effect, says that one may restrict attention to a specific sequence h_n:

The Riemann Hypothesis is true if and only if $\lambda_n \ge 0$ for each $n = 1, 2, \ldots$ where

$$\lambda_n = \sum_\rho (1 - (1 - 1/\rho)^n).$$

As usual, the sum over zeros is $\lim_{T\to\infty}\sum_{|\rho|<T}$. Another expression for λ_n is

$$\lambda_n = \frac{1}{(n-1)!}\,\frac{d^n}{ds^n}(s^{n-1}\log\xi(s))\Big|_{s=1}.$$

It would be interesting to find an interpretation (geometric?) for these λ_n, or perhaps those associated with a different L-function, to make their positivity transparent.

Selberg's Trace Formula

Selberg, perhaps looking for a spectral interpretation of the zeros of $\zeta(s)$, proved a trace formula for the Laplace operator acting on the space of real-analytic functions defined on the upper half-plane $\mathcal{H} = \{x + iy : y > 0\}$ and invariant under the group $SL(2,\mathbb{Z})$ of linear fractional transformations with integer entries and determinant one, which acts discontinuously on $\mathcal{H}$. This invariance is expressed as

$$f\left(\frac{az+b}{cz+d}\right) = f(z);$$

the Laplace operator in this case is

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right).$$

The spectrum of Δ splits into a continuous part and a discrete part. The eigenvalues λ are all positive and, by convention, are usually expressed as $\lambda = s(1-s)$. The continuous part consists of all $s = 1/2 + it$, $t \ge 0$, and we write the discrete part as $s_j = \frac{1}{2} + ir_j$. Then

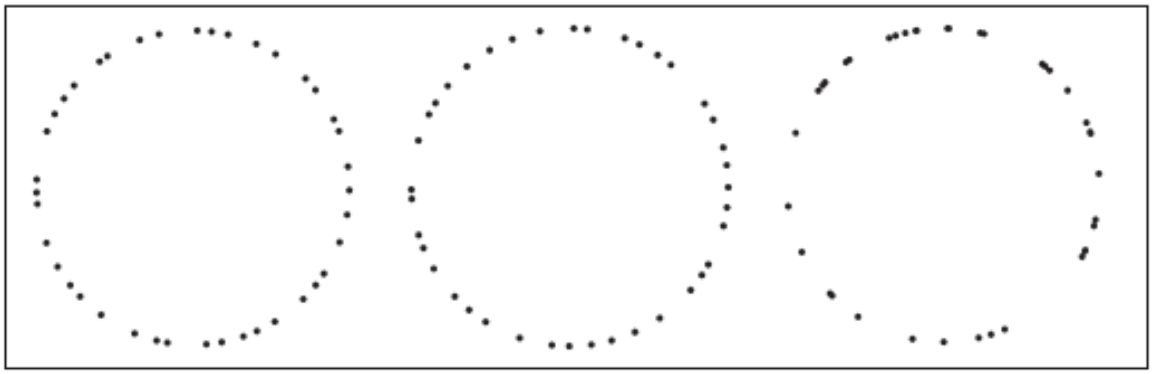


Figure 7. The eigenvalues of a random 40 x 40 unitary matrix, 40 consecutive zeros of $\zeta(s)$ scaled to wrap once around the circle, and 40 randomly chosen points on the unit circle.

$$\sum_{j=1}^{\infty} h(r_j) = -h(0) - g(0)\log\tfrac{\pi}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)G(r)\,dr + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n) + \sum_{P}\sum_{\ell=1}^{\infty}\frac{g(\ell\log P)\log P}{P^{\ell/2}-P^{-\ell/2}}$$

where g, h, and Λ are as in Weil's formula and

$$G(r) = \frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir) - \frac{\pi}{6}r\tanh\pi r + \frac{\pi}{\cosh\pi r}(\tfrac{1}{8}+\tfrac{\sqrt{3}}{9}\cosh\tfrac{\pi r}{3}).$$

The final sum is over the norms P of prime geodesics of $SL(2,\mathbb{Z})\backslash\mathcal{H}$. The values taken on by P are of the form $(n+\sqrt{n^2-4})^2/4$, $n \geq 3$, with certain multiplicities (the class number $h(n^2-4)$). H. Haas was one of the first people to compute the eigenvalues $r_1 = 9.533\ldots$, $r_2 = 12.173\ldots$, $r_3 = 13.779\ldots$ of $SL(2,\mathbb{Z})$ in 1977 in his University of Heidelberg Diplomarbeit. Soon after, Hejhal was visiting San Diego, and Audrey Terras pointed out to him that Haas's list contained the numbers $14.134\ldots, 21.022\ldots$: the ordinates of the first few zeros of $\zeta(s)$ were lurking amongst the eigenvalues! Hejhal discovered the ordinates of the zeros of $L(s,\chi_3)$ (see section 7) on the list too. He unraveled this perplexing mystery about six months later. It turned out that the spurious eigenvalues were associated to "pseudo cusp forms" and appeared because of the method of computation used. If the zeros had appeared legitimately, RH would have followed because $\lambda = \rho(1-\rho)$ is positive. (The 1979 IHÉS preprint by P. Cartier and Hejhal contains additional details of the story.)

The trace formula resembles the explicit formula in certain ways. Many researchers have attempted to interpret Weil's explicit formula in terms of Selberg's trace formula.

Some Other Equivalences of Interest

Here are a few other easy-to-state equivalences of RH:

- Hardy and Littlewood (1918): *RH holds if and only if*
$$\sum_{k=1}^{\infty}\frac{(-x)^k}{k!\,\zeta(2k+1)} = O(x^{-1/4}) \quad \text{as } x\to\infty.$$
- Redheffer (1977): *RH holds if and only if for every $\epsilon > 0$ there is a $C(\epsilon) > 0$ such that $|\det(A(n))| < C(\epsilon)n^{1/2+\epsilon}$, where $A(n)$ is the $n\times n$ matrix of 0's and 1's defined by $A(i,j) = 1$ if $j = 1$ or if i divides j, and $A(i,j) = 0$ otherwise.* It is known that $A(n)$ has $n - [n\log 2] - 1$ eigenvalues equal to 1. Also, A has a real eigenvalue (the spectral radius) which is approximately $\sqrt{n}$, a negative eigenvalue which is approximately $-\sqrt{n}$, and the remaining eigenvalues are small.
- Lagarias (2002): Let $\sigma(n)$ denote the sum of the positive divisors of n. *RH holds if and only if*
$$\sigma(n) \leq H_n + \exp(H_n)\log H_n$$
for every n, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Other Zeta- and L-Functions

Over the years striking analogies have been observed between the Riemann zeta-function and other zeta- or L-functions. While these functions are seemingly independent of each other, there is growing evidence that they are all somehow connected in a way that we do not fully understand. In any event, trying to understand, or at least classify, all of the objects which we believe satisfy RH is a reasonable thing to do. The rest of the article will give a glimpse in this direction and perhaps a clue to the future.

First, some examples of other functions that we believe satisfy RH. The simplest after ζ is the Dirichlet L-function for the nontrivial character of conductor 3:

$$L(s,\chi_3) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \ldots.$$

This can be written as an Euler product

$$\prod_{p\equiv 1 \bmod 3} (1-p^{-s})^{-1} \prod_{p\equiv 2 \bmod 3} (1+p^{-s})^{-1},$$

it satisfies the functional equation

$$\xi(s,\chi_3) := \left(\frac{\pi}{3}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi_3) = \xi(1-s,\chi_3),$$

and it is expected to have all of its nontrivial zeros on the $1/2$-line. A similar construction works for any primitive Dirichlet character.

Dedekind, Hecke, Artin, and others developed the theory of zeta-functions associated with number fields and their characters. These have functional equations and Euler products, and are expected to satisfy a Riemann Hypothesis. Ramanujan's tau-function defined implicitly by

$$x\prod_{n=1}^{\infty}(1-x^n)^{24} = \sum_{n=1}^{\infty} \tau(n)x^n$$

also yields an L-function. The associated Fourier series $\Delta(z) := \sum_{n=1}^{\infty} \tau(n)\exp(2\pi i n z)$ satisfies

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for all integers a, b, c, d with $ad - bc = 1$. A function satisfying these equations is called a *modular form* of weight 12. The associated L-function

$$L_\Delta(s) := \sum_{n=1}^{\infty} \frac{\tau(n)/n^{11/2}}{n^s} = \prod_p \left(1 - \frac{\tau(p)/p^{11/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

satisfies the functional equation

$$\xi_\Delta := (2\pi)^{-s}\Gamma(s+11/2)L_\Delta(s) = \xi_\Delta(1-s),$$

and all of its complex zeros are expected to be on the $1/2$-line.

Another example is the L-function associated to an elliptic curve $E: y^2 = x^3 + Ax + B$, where A and B are integers. The associated L-function, called the Hasse-Weil L-function, is

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a(n)/n^{1/2}}{n^s} = \prod_{p\nmid N}\left(1 - \frac{a(p)/p^{1/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \times \prod_{p\mid N}\left(1 - \frac{a(p)/p^{1/2}}{p^s}\right)^{-1},$$

where N is the conductor of the curve. The coefficients a_n are constructed easily from a_p for prime p; in turn the a_p are given by $a_p = p - N_p$, where N_p is the number of solutions of E when considered modulo p. The work of Wiles and others proved that these L-functions are associated to modular forms of weight 2. This modularity implies the functional equation

$$\xi_E(s) := (2\pi/\sqrt{N})^{-s}\Gamma(s+1/2)L_E(s) = \xi_E(1-s).$$

It is believed that all of the complex zeros of $L_E(s)$ are on the $1/2$-line. A similar construction ought to work for other sets of polynomial equations, but so far this has not been proved.

What is the most general situation in which we expect the Riemann Hypothesis to hold? The Langlands program is an attempt to understand all L-functions and to relate them to automorphic forms. At the very least a Dirichlet series that is a candidate for RH must have an Euler product and a functional equation of the right shape. Selberg has given a set of four precise axioms which are believed to characterize the L-functions for which RH holds. Examples have been given that show the necessity of most of the conditions in his axioms.

L-Functions and Random Matrix Theory

An area of investigation which has stimulated much recent work is the connection between the Riemann zeta-function and Random Matrix Theory (RMT). This work does not seem to be leading in the direction of a proof of RH, but it is convincing evidence that the spectral interpretation of the zeros sought by Hilbert and Pólya is an idea with merit. Moreover, the connection between zeta theory and RMT has resulted in a very detailed model of $\zeta(s)$ and its value distribution.

Montgomery's Pair Correlation Conjecture

In 1972 Hugh Montgomery was investigating the spacings between zeros of the zeta-function in an attempt to solve the class number problem. He formulated his Pair Correlation Conjecture based in part on what he could prove assuming RH and in part on RH plus conjectures for the distribution of twin primes and other prime pairs. This conjecture asserts that

$$\sum_{\frac{2\pi\alpha}{\log T} < \gamma-\gamma' \le \frac{2\pi\beta}{\log T}} 1 \sim N(T)\int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du.$$

The sum on the left counts the number of pairs $0 < \gamma, \gamma' < T$ of ordinates of zeros with normalized spacing between positive numbers $\alpha < \beta$. Montgomery had stopped in Princeton on his way from St. Louis, where he had presented this result at an AMS symposium, to Cambridge University, where he was a graduate student. Chowla persuaded him to show this result to Freeman Dyson at afternoon tea at the Institute for Advanced Study. Dyson immediately identified the integrand $1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$ as the pair correlation function for eigenvalues of large random Hermitian matrices measured with a Gaussian measure—the Gaussian Unitary Ensemble that physicists had long been studying

in connection with the distribution of energy levels in large systems of particles. With this insight, Montgomery went on to conjecture that perhaps all the statistics, not just the pair correlation statistic, would match up for zeta-zeros and eigenvalues of Hermitian matrices. This conjecture is called the GUE conjecture. It has the flavor of a spectral interpretation of the zeros, though it gives no indication of what the particular operator is.

Odlyzko's Calculations

In the 1980s Odlyzko began an intensive numerical study of the statistics of the zeros of $\zeta(s)$. Based on a new algorithm developed by Odlyzko and Schönhage that allowed them to compute a value of $\zeta(1/2 + it)$ in an average time of t^{ϵ} steps, he computed millions of zeros at heights around 10^{20} and spectacularly confirmed the GUE conjecture.

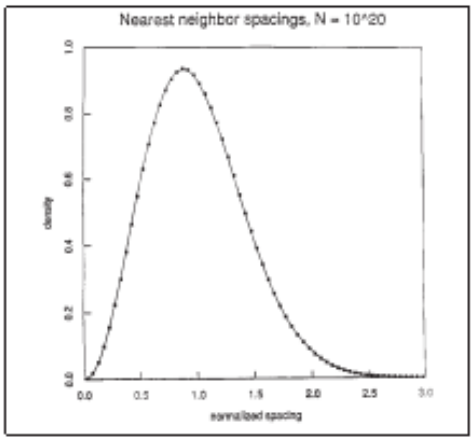


Figure 8a. The nearest neighbor spacing for GUE (solid) and for 7.8×10^7 zeros of $\zeta(s)$ near the 10^{20} zero (scatterplot). Graphic by A. Odlyzko.

Moments of Zeta

More recently, RMT has led to a conjecture for moments of ζ on the critical line. Let

$$I_k(T) = \frac{1}{T}\int_0^T |\zeta(1/2 + it)|^{2k}\, dt.$$

Asymptotic formulas for I_1 and I_2 were found by Hardy and Littlewood and Ingham by 1926. In 1995 Ghosh and I formulated a conjecture for I_3 and set up a notation to clarify the part missing from our understanding of I_k. After scaling out the arithmetic parts, we identified a factor g_k which we could not predict. The factor is $g_1 = 1$ and $g_2 = 2$ for the second and fourth moments and conjecturally $g_3 = 42$ for the sixth moment. At RHI in Seattle, Sarnak proposed to Keating that he find a random matrix explanation for these numbers. By 1998 Gonek and I had found a number-theoretic way to conjecture the answer for the eighth moment, namely $g_4 = 24024$. At RHII in Vienna, Keating announced that he and Snaith had a conjecture for all of the moments which agreed with g_1, g_2, and g_3. Keating, Snaith, and I—moments before Keating's lecture—checked (amid great excitement!) that the Keating and Snaith conjecture also produced $g_4 = 24024$.

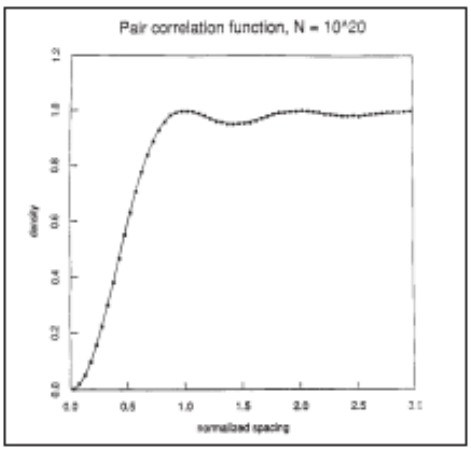


Figure 8b.The pair-correlation function for GUE (solid) and for 8×10^6 zeros of $\zeta(s)$ near the 10^{20} zero (scatterplot). Graphic by A. Odlyzko.

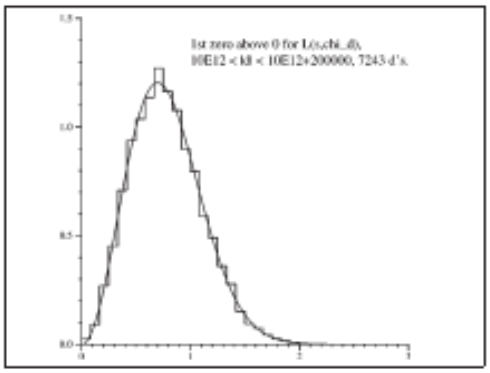
The idea of Keating and Snaith was that if the eigenvalues of unitary matrices model zeta zeros, then perhaps the characteristic polynomials of unitary matrices model zeta values. They were able to compute—exactly—the moments of the characteristic polynomials of unitary matrices averaged with respect to Haar measure by using Selberg's integral, which is a formula found in the 1940s by Selberg that vastly generalizes the integral for the beta-function. Keating and Snaith proposed that

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Farmer and I (2000) proved that g_k is always an integer and found that it has an interesting prime factorization.

Families

At RHI in Seattle, Sarnak gave a lecture on families of L-functions based on work that he and Katz were doing. They discovered a way to identify a symmetry type (unitary, orthogonal, or symplectic) with various families of L-functions. Their work was based on studying families of zeta-functions over finite fields (for which RH was already proved by Weil for curves and by Deligne for general varieties). For these zeta-functions, Katz and Sarnak proved that the zeros of the family were distributed exactly



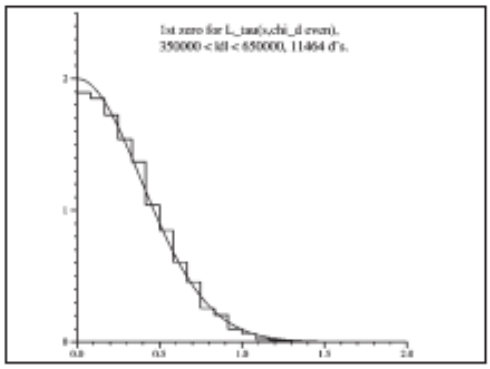


Figure 9. A comparison of the distribution of the lowest lying zero for two families of L-functions. In each case one first needs to suitably normalize the zeros. The first figure compares the distribution of the lowest zero of $L(s, \chi_d)$, Dirichlet L-functions, for several thousand d's of size 10^{12}, against the distribution of the zero closest to 1 for large unitary symplectic matrices. In the second picture we show the same statistic, but for several thousand even quadratic twists d of size 500,000, of the Ramanujan τ cusp form L-function. This is compared to the distribution of the zero closest to 1 for large orthogonal matrices with even characteristic polynomial (in the latter family, one needs to distinguish between even and odd twists). Graphics by M. Rubinstein.

as the RMT distributions of the monodromy group associated with the family.

Katz and Sarnak stress that the proofs of Weil and Deligne use families of zeta-functions over finite fields to prove RH for an individual zeta-function. The modelling of families of L-functions by ensembles of random matrix theory gives evidence for a spectral interpretation of the zeros, which may prove important if families are ultimately used to prove RH. At this point, however, we do not know what plays the role of the monodromy groups in this situation.

RMT and Families

Keating and Snaith extended their conjectures to moments of families of L-functions by computing moments of characteristic polynomials of symplectic and orthogonal matrices, each with their own Haar measure. (It should be mentioned that the orthogonal and symplectic circular ensembles used by the physicists do not use Haar measure and so have different answers. Katz and Sarnak figured out that Haar measure must be used to model L-functions.)

Further works by Farmer, Keating, Rubinstein, Snaith, and this author have led to precise conjectures for all of the main terms in moments for many families of L-functions. These results are so precise that they lead to further conjectures about the distribution of values of the L-functions. We can even predict how frequently we find double zeros at the center of the critical strip of L-functions within certain families.

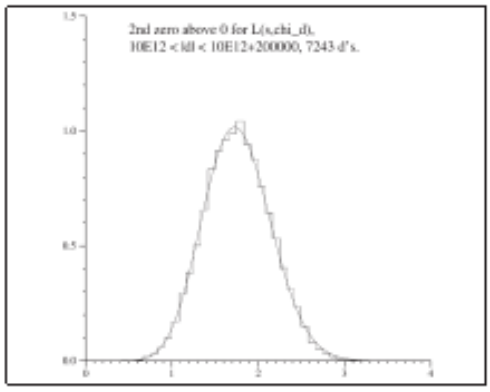


Figure 10. The second zero for $L(s, \chi_d)$ as compared to the RMT prediction. Graphic by M. Rubinstein.

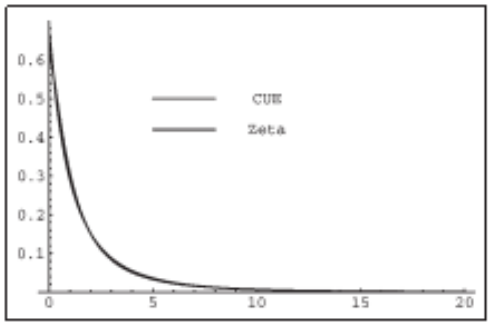


Figure 11. The distribution of values of $|\zeta(1/2 + it)|$ near $t = 10^6$ compared with the distribution of values of characteristic polynomials of 12×12 unitary matrices. Graphic by N. Snaith.

The Conspiracy of L-Functions

There is a growing body of evidence that there is a conspiracy among L-functions—a conspiracy which is preventing us from solving RH!

The first clue that zeta- and L-functions even know about each other appears perhaps in works of Deuring and Heilbronn in their study of one of the most intriguing problems in all of mathematics: Gauss's class number problem. Gauss asked whether the number of equivalence classes of binary quadratic forms of discriminant $d < 0$ goes to ∞ as d goes to $-\infty$.

The equivalence class of a quadratic form $Q(m,n) = am^2 + bmn + cn^2$ of discriminant $d = b^2 - 4ac$ consists of all of the quadratic forms obtained by a linear substitution $m \to \alpha m + \beta n$, $n \to \gamma m + \delta n$, where $\alpha, \beta, \gamma, \delta$ are integers with $\alpha\delta - \beta\gamma = 1$. The number $h(d)$ of these equivalence classes is called the class number and is known to be finite. Equivalently, $h(d)$ is the number of ideal classes of the imaginary quadratic field $Q(\sqrt{d})$. The history of Gauss's problem is extremely interesting; it has many twists and turns and is not yet finished—we seem to be players in the middle of a mystery novel.

Deuring and Heilbronn were trying to solve Gauss's problem. The main tool they were using was the beautiful class number formula of Dirichlet, $h(d) = \sqrt{|d|}\, L(1,\chi_d)/\pi$ $(|d| > 4)$, which gives the class number in terms of the value of the L-function at 1, which is at the edge of the critical strip. So the question boils down to giving a lower bound for $L(1,\chi_d)$; this question, in turn, can be resolved by proving that there is no real zero of $L(s,\chi_d)$ very near to 1.

Hecke had shown that the truth of RH for $L(s,\chi_d)$ implies that $h(d) \to \infty$. Then Deuring proved that the falsity of RH for $\zeta(s)$ implies that $h(d) > 1$ for large $|d|$. Finally, Heilbronn showed that the falsity of RH for $L(s,\chi)$ for any χ implied that $h(d) \to \infty$. These results together proved Gauss's conjecture and gave a first indication of a connection between the zeros of $\zeta(s)$ and those of $L(s,\chi_d)$!

Later Landau showed that a hypothetical zero of $L(s,\chi_{d_1})$ very near to 1 implies that no other $L(s,\chi_d)$, $d \neq d_1$, could have such a zero, further illustrating that zeros of $L(s,\chi_d)$ know about each other. Siegel strengthened this approach to show that for every $\epsilon > 0$ there is a $c(\epsilon) > 0$ such that no zero β of $L(s,\chi_d)$ satisfies $\beta > 1 - c(\epsilon)|d|^{-\epsilon}$. The problem with the arguments of Landau and Siegel is that the constant $c(\epsilon)$ cannot be effectively computed, and so the bound cannot be used to actually calculate the list of discriminants d with a given class number, which presumably is what Gauss wanted. The ineffectivity comes about from the assumption that some L-function actually has a real zero near 1. Such a hypothetical zero of some L-function, which no one believes exists, is called a Landau-Siegel zero.

In fact, one can show that if there is some d_1 such that $L(s,\chi_{d_1})$ has a zero at $\beta < 1$, then it follows that $h(d) > c|d|^{\beta-1/2}/\log|d|$ for all other d, where $c > 0$ can be effectively computed. Thus, the closer to 1 the hypothetical zero is, the stronger the result. But note also that any zero bigger than 1/2 would give a result. The basic idea behind this approach is that if there is an $L(s,\chi_d)$ with a zero near 1, then $\chi_d(p) = -1$ for many small primes. In other words, χ_d mimics the Möbius function $\mu(n)$ for small n. This is consistent with the fact that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

has a zero at $s = 1$ (since $\zeta(s)$ has a pole at $s = 1$).

The Landau-Siegel Zero

Much effort has gone toward trying to eliminate the Landau-Siegel zero described above and so find an effective solution to Gauss's problem. However, the L-function conspiracy blocks every attempt exactly at the point where success appears to be in sight. We begin to suspect that the battle for RH will not be won without getting to the bottom of this conspiracy. Here are some tangible examples which give a glimpse of this tangled web.

The Brun-Titchmarsh theorem. Let $\pi(x;q,a)$ denote the number of primes less than or equal to x that lie in the arithmetic progression $a \bmod q$. Sieve methods can show that for any $1 \le q < x$ the inequality

$$\pi(x;q,a) \le 2\frac{x}{\phi(q)\log(x/q)}$$

holds, where ϕ is Euler's phi-function. It is believed that the same theorem should be true with 2 replaced by any number larger than 1 and sufficiently large x. Any lowering of the constant 2 would eliminate the Landau-Siegel zero. In particular, Motohashi [1979] proved that if $1-\delta$ is a real zero of $L(s,\chi_q)$, then if for $x \ge q^c$ the Brun-Titchmarsh theorem is valid in the form $\pi(x;q,a) \le (2-\alpha)x/(\phi(q)\log(x/q))$, where $\alpha > 0$ is an absolute constant, then $\delta \ge c'\xi/\log q$, where c and c' are certain numerical constants.

The Alternative Hypothesis. This is an alternative to the GUE model for the distribution of zeros. It proposes the existence of a function $f(T)$ that goes to 0 as $T \to \infty$ such that if any two consecutive ordinates γ and γ' of zeros of ζ larger than some T_0 are given, then the normalized gap $2\pi(\gamma\log\gamma - \gamma'\log\gamma')$ between γ and γ' is within $f(T_0)$ of half of an integer. This hypothesis is clearly absurd! However, ruling this out would eliminate the Landau-Siegel zero (Conrey-Iwaniec (2002)), and so for all we know it could be true.

If one could prove, for example, that there is a $\delta > 0$ such that for all sufficiently large T there is a pair of consecutive zeros with ordinates between T and $2T$ whose distance apart is less than $1/2 - \delta$ times the average spacing, then the alternative hypothesis would be violated. Random matrix theory predicts the exact distribution of these neighbor spacings and shows that we should expect that about 11 percent of the time the neighbor gaps are smaller than $1/2$ of the average. These ideas were what led Montgomery to consider the pair-correlation of the zeros of $\zeta(s)$ mentioned above. He showed that there are arbitrarily large pairs of zeros that are as close together as 0.68 of the average spacing. Later works have gotten this bound down to 0.5152. There are indications that using work of Rudnick and Sarnak on higher correlations of the zeros of ζ, one might be able to reach 0.5, but 0.5 is definitely a limit (more like a brick wall!) of all of the known methods.

Vanishing of modular L-functions. The most spectacular example is the work of Iwaniec and Sarnak. They showed that if one could prove that there is a $\delta > 0$ such that more than $1/2 + \delta$ of the modular L-functions of a fixed weight, large level, and even functional equation do not vanish, then the Landau-Siegel zero could be eliminated. It is predicted that all but an infinitesimal proportion of these values are nonzero; they just needed one-half plus δ of them to be nonzero. They can prove that 50 percent do not vanish, but despite their best efforts they cannot get that extra little tiny bit needed to eliminate the Landau-Siegel zero.

A Clue and a Partial Victory

The only approach that has made an impact on the Landau-Siegel zero problem is an idea of Goldfeld. In 1974 Goldfeld, anticipated somewhat by Friedlander, realized that while a zero at $1/2$ would barely fail to produce a lower bound for the class number tending to infinity, a multiple zero at $1/2$ would produce a lower bound which, while not a positive power of $|d|$, still goes to ∞. Moreover, it was believed—by virtue of the Birch and Swinnerton-Dyer conjecture—that zeros of high multiplicity do exist and the place to look for them is among L-functions associated to elliptic curves with large rank. However, it was not until 1985 that Gross and Zagier demonstrated conclusively that there exist L-functions with triple zeros at $1/2$. This led to the lower bound that for any $\epsilon > 0$ there is an effectively computable $c_1(\epsilon) > 0$ such that $h(d) > c_1(\epsilon)(\log|d|)^{1-\epsilon}$. This is a long way from the expected $h(d) > c\sqrt{|d|}/\log|d|$, but it did solve Gauss's problem. The clue that it gave us was to study exotic L-functions, or extremal L-functions, which have zeros of high multiplicity at the center. At present, our best hope for finding these L-functions is to look at elliptic curves with many rational points.

Iwaniec's Approach

Iwaniec, in his lecture at RHIII, proposed a way to take advantage of the above ideas. In a nutshell, his idea is to take a family of L-functions having a multiple zero at $1/2$ and use this family to obtain useful approximations for the Möbius function $\mu(n)$ as a linear combination of the coefficients of the L-functions from the family. In this way, the Möbius function is tamed. One example of a family considered by Iwaniec is the family of L-functions associated to the elliptic curves

$$E_{A,B^2} : y^2 = x^3 + Ax + B^2,$$

which have a rational point $(B, 0)$ and so have rank at least one. Considering A and B in certain arithmetic progressions shows that the associated L-function must have a double zero at the center.

Iwaniec presented three conjectures which together would eliminate the Landau-Siegel zero. The main two theorems needed to complete his program are a bound for the second moment

$$\sum_{A \approx X^{1/2}, B \approx X^{1/4}} L_{A,B^2}(1/2)^2 = O\left(X^{7/12}(\log X)^C\right)$$

of this family together with a good estimate (square-root cancellation uniform in M, N, and q) for the incomplete exponential sum

$$\sum_{M<m<2M, N<n<2N} \chi_q(mn) \exp\left(2\pi i \frac{m^3 n^{-4}}{q}\right),$$

the kind of estimate that for a completed exponential sum follows from the RH for varieties proved by Deligne. Iwaniec has similar, but more complicated, constructions that would lead to a quasi-Riemann hypothesis, producing a concrete $\beta < 1$ such that there are no zeros to the right of the line through β.

Iwaniec's approach will likely reduce the question of RH, which is ostensibly about zeros or poles, into several subsidiary questions that have a much different flavor, such as finding upper bound estimates for moments and values of L-functions. This approach offers hope of attack by methods from analytic number theory.

Conclusion

A major difficulty in trying to construct a proof of RH through analysis is that the zeros of L-functions behave so much differently from zeros of many of the special functions we are used to seeing in mathematics and mathematical physics. For example, it is known that the zeta-function does not satisfy any differential equation. The functions which do arise as solutions of some of the classical differential equations, such as Bessel functions, hypergeometric

functions, etc., have zeros which are fairly regularly spaced. A similar remark holds for the zeros of solutions of classical differential equations regarded as a function of a parameter in the differential equation. For instance, in the Pólya theorem above comparing $\phi(t)$ with $\Phi(t)$, the zeros are actually zeros of a Bessel function of fixed argument regarded as a function of the index. Again the zeros are regularly spaced.

On the other hand, the zeros of L-functions are much more irregularly spaced. For example, the RMT models predict that for any $\epsilon > 0$ there are infinitely many pairs of zeros ρ and ρ' such that $|\rho - \rho'| < |\rho|^{-1/3+\epsilon}$. Generally it is believed that all zeros of all L-functions are linearly independent (in particular, simple), except that certain L-functions can have a zero at $s = 1/2$ of high multiplicity. The conjecture of Birch and Swinnerton-Dyer asserts that the multiplicity of the zero of the L-function associated with a given elliptic curve is equal to the rank of the group of rational points on the elliptic curve. It is known that the latter can be as large as 26, and it is generally believed to get arbitrarily large. None of the methods from analysis seem capable of dealing with such exotic phenomena.

It is my belief that RH is a genuinely arithmetic question that likely will not succumb to methods of analysis. There is a growing body of evidence indicating that one needs to consider families of L-functions in order to make progress on this difficult question. If so, then number theorists are on the right track to an eventual proof of RH, but we are still lacking many of the tools. The ingredients for a proof of RH may well be moment theorems for a new family of L-functions not yet explored; modularity of Hasse-Weil L-functions for many varieties, like that proved by Wiles and others for elliptic curves; and new estimates for exponential sums, which could come out of arithmetic geometry. The study of L-functions is still in its beginning stages. We only recently learned the modularity of the L-functions associated to elliptic curves; it would be very helpful to understand the L-functions for more complicated curves and generally for varieties. It would be useful to systematically compute many new examples of L-functions to get a glimpse of what is out there waiting to be discovered. The exotic behavior of the multiple zeros of L-functions associated to elliptic curves with many rational points could be just the beginning of the story.

Acknowledgements

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—*J.B.C.*

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Notes

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