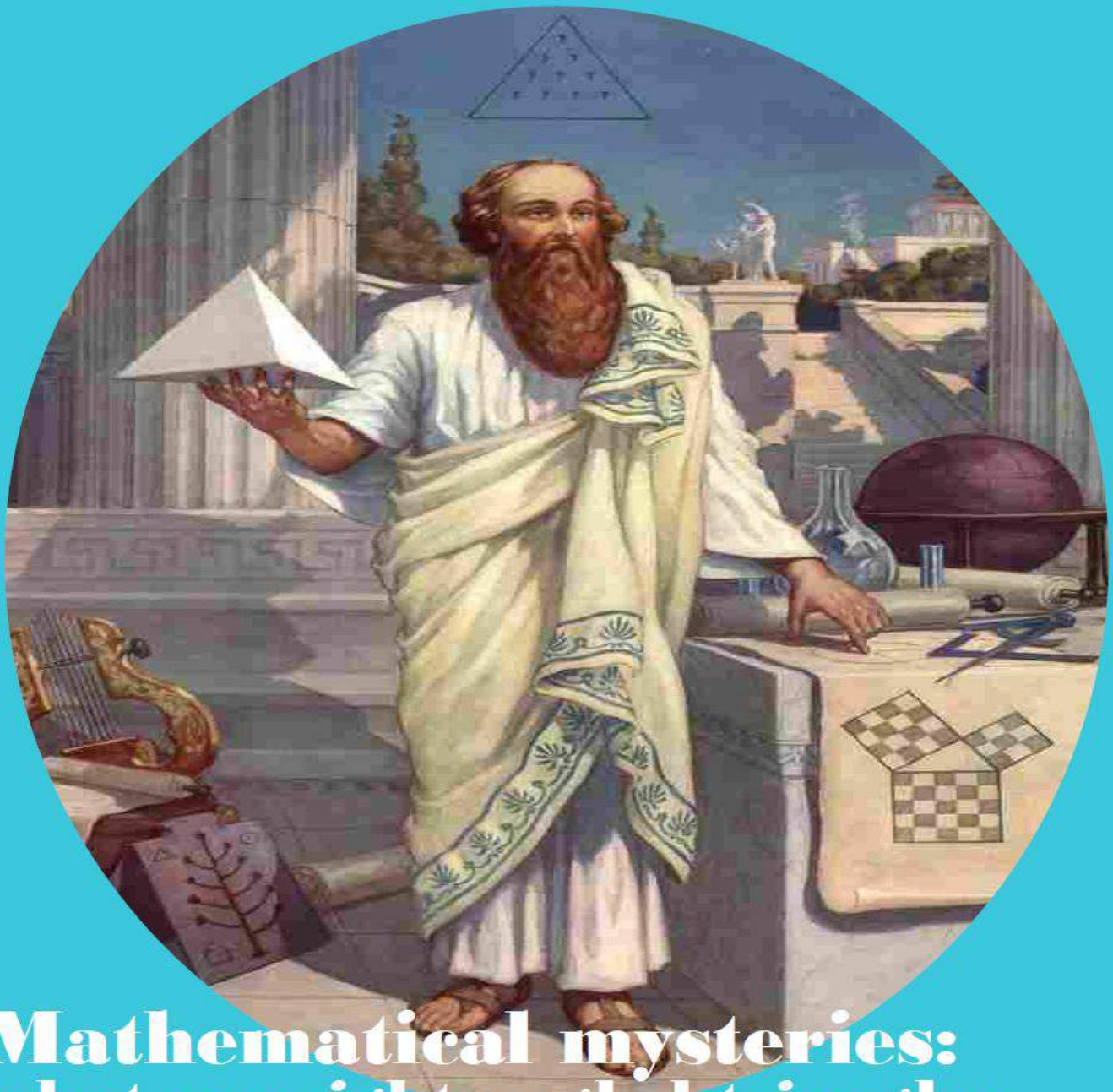

Mathematical mysteries: what are right-angled triangles hiding from you?

Key Stage 4 Programme

Pupil Name	
Handbook Designed by	Dr. Danny Scarponi

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1	**12th December 2018**	**2.00 – 3.00 pm**	**King's College**
2			**Harris Academy Beckenham**
3			**Harris Academy Beckenham**
4			**Harris Academy Beckenham**
5			**Harris Academy Beckenham**
6 (Feedback)			**Harris Academy Beckenham**
7 (Feedback)			**Harris Academy Beckenham**

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1	Pythagoreanism and Pythagoras' Theorem	
Tutorial 2	Hippasus of Metapontum and irrational numbers	
Tutorial 3	Fibonacci and congruent numbers	
Tutorial 4	Pierre de Fermat and Diophantine equations	
Tutorial 5	History of Congruent Number Problem and related exercises	

Assignment Submission – Lateness and Plagiarism

Lateness	
Submission after midnight on 9th August	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Uni Pathways – Pupil Feedback Report

Grade	Marks	Guidance
1st	70+	Performing to an excellent standard at A-level
2:1	60-69	Performing to a good standard at A-level
2:2	50-59	Performing to an excellent standard at GCSE
3rd	40-49	Performing to a good standard at GCSE
Working towards a pass	0-39	Performing below a good standard at GCSE
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Name of PhD Tutor	Dr. Danny Scarponi		
Title of Assignment	Final Assignment		
Name of Pupil			
Name of School	Harris Academy Beckenham		
ORIGINAL MARK / 100		FINAL MARK / 100	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the PhD tutor should give an explanation in this section:

Learning Feedback Comment 1 - ***Enter Key Learning Priority Here***	
What you did in relation to this Key Learning Priority *Enter feedback here*	How you could improve in the future *Enter feedback here*
Learning Feedback Comment 2 – ***Enter Key Learning Priority Here***	
What you did in relation to this Key Learning Priority *Enter feedback here*	How you could improve in the future *Enter feedback here*
Learning Feedback Comment 3 – ***Enter Key Learning Priority Here***	
What you did in relation to this Key Learning Priority *Enter feedback here*	How you could improve in the future *Enter feedback here*
Resilience Comment	
How you showed learning resilience during the course *Enter feedback here*	How you could build learning resilience in the future *Enter feedback here*

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Course Rationale

In mathematics, a congruent number is a positive integer that is the area of a right triangle with three rational sides. The question of determining whether a positive integer is a congruent number is called the Congruent Number Problem. Already investigated by the Persian mathematician al-Karaji in the 10th century, this problem has not (as of 2018) been brought to a successful resolution. Many great minds, such as Leonardo Fibonacci, Pierre de Fermat and J. B Tunnell have studied it and have given their own contribution, proving some partial results that help us understand the complexity of congruent numbers.

The Congruent Number Problem is a perfect example of “Diophantine problem”, i.e. a problem which becomes very easy to solve once one forgets the rationality condition. In other terms, the Congruent Number Problem is extremely difficult to solve because it asks us not just to find a right triangle but a right triangle with rational sides.

Why present this problem to high school pupils? First of all, because it is a piece of beautiful and powerful mathematics, at the same time very old and of extreme interest to the arithmetic geometers of our days. Secondly, the topic will be treated with a particular attention to its historical development: our aim is for the students to familiarize themselves with the idea that mathematics, like any other science, is a process in continuous evolution. This is probably quite different from the understanding of mathematics that pupils form just based on the high school curriculum. Pupils will get accustomed with the fundamental concepts of theorem, proof, open problem and partial solution of a problem. Sketches of two short proofs will be presented to them and they will be asked to write the proofs in detail as homework. This activity will give them the possibility to learn to use mathematical vocabulary and notation correctly and to write texts in which every assertion follows from the previous in a clear and logical way. Hopefully, pupils will get excited by the discovery that still nowadays, we have millions of mathematical questions that we don’t have any idea how to solve. Mathematics is more than ever in need of new brilliant minds!

Here is a brief overview of the course: we will start learning about Pythagoras’ theorem and rational/irrational numbers (Tutorials 1 and 2). When these concepts are familiar to the pupils, we will introduce the Congruent Number Problem (Tutorial 3). We will produce some examples of congruent numbers using Pythagorean triples and a result from Paul Monsky (Tutorial 4). We will comment on Fermat’s Right Triangle Theorem and on “Tunnell’s Theorem” and we will use them to show that certain numbers are not congruent (Tutorial 5).

Mark Scheme

STEM Mark Scheme

Skills	1st (70-100)	2:1 (60-69)	2:2 (50-59)
Knowledge and Understanding	• All content included is relevant to the general topic and to the specific question/title • Good understanding of all the relevant topics. • Scientific terms are defined and used accurately throughout • Clear justification on how the content included is related to the specific issues that are the focus of the assignment	• Most of the content included is relevant to the general topic and to the specific question/title • Good understanding of most the relevant topics • Scientific terms are used accurately but not always clearly defined. • Adequate justification on how the content included is related to the specific issues that are the focus of the assignment	• Some of the content included is relevant to the general topic and to the specific question/title • Good understanding on some of the relevant topics but occasional confusion on others. • Scientific terms are used mostly accurately with occasional confusion and often not defined. • Some justification on how the content included is related to the specific issues that are the focus of the assignment
Critical Evaluation / Problem solving	• Moved beyond description to an assessment of the value or significance of what is described • Evaluative points are consistently explicit/systematic/reasoned/justified • Effective critiques on the reliability of sources provided • Consistently demonstrate clear, analytical and logical steps to solving problems	• Mostly description but some assessment of the value or significance of what is described • Evaluative points are mostly explicit/systematic/reasoned/justified • Some evidence of critiques on the reliability of sources provided • Some examples of solving problems but not consistently clear, analytical and logical	• Only description with minimal assessment of the value or significance of what is described • Evaluative points are at times explicit/systematic/reasoned/justified • Limited evidence of critiques on the reliability of sources provided • Demonstrate clear steps to solving problems but not consistently analytical or logical

Structure and Presentation	• Ideas are presented in paragraphs and arranged in a logical structure that is appropriate for the assignment • The introduction clearly outlines how the essay/report will deal with the issues • The conclusion summarises all the main points clearly and concisely • All calculations, formulas and methods are clearly structured, clear to follow and correct • Tables and graphs are effectively constructed including appropriate headings, units and scales. • All sources are referenced correctly in an agreed format	• Ideas are presented in paragraphs and arranged in a structure that is mostly appropriate for the assignment • The introduction adequately describes how the essay/report will deal with the issues • The conclusion summarises most of the main points clearly • Calculations, formulas and methods are mostly structured, clear to follow and correct • Most tables and graphs are well constructed • Most sources are referenced correctly in an agreed format	• Ideas are presented in paragraphs and arranged in a structure • The introduction mentions how the essay/report will deal with the issues • The conclusion summarises some of the main points clearly • Calculations, formulas and methods are not always structured, clear to follow and correct. • Some tables and graphs are well constructed but contains some errors • Some sources are referenced correctly in the agreed format with occasional errors

Subject Vocabulary

The first row is already filled. You will fill the rest of the table as we discover and learn these new words in the tutorials.

Word	Definition	In a sentence
Axiom	An axiom is a statement that is taken to be true, to serve as a starting point for further reasoning and arguments.	Euclid gave five axioms in his book on geometry “The Elements”. One of these is the well-known “Parallel axiom”.
Theorem		
Proof		
Integer		
Pythagorean triples		
Rational number		
Irrational number		
Congruent number		
Partial solution		
Prime number		
Diophantine problem		
Rational right triangle		

Knowledge Organizers

Important vocabulary	
Rational number	A number that can be expressed as the fraction of two integers
Irrational number	A number that is not rational.
Theorem	A proven idea in mathematics.
Rational right triangle	A right-angled triangle with three rational sides.
Congruent number	A positive integer that is the area of a rational right triangle.
Diophantine equation	A polynomial equation such that only the integer solutions are sought.
Open problem	A (mathematical) problem for which we have not found a solution yet.

Decimal expansion of rational/irrational numbers	
Rational	Its decimal expansion is either finite or infinite but periodic (eventually with an anti-period)
Irrational	Its decimal expansion is infinite and not periodic

Key facts about the Congruent Number Problem	
What we do not know yet	The Congruent Number Problem is still open, i.e. we do not have a general rule to decide whether a positive integer is congruent.
What we know	1. Pythagorean triples give some examples of congruent numbers. 2. Fermat's Right Triangle Theorem: the first result proving that not all numbers are congruent. 2. Paul Monsky's Theorems: prime numbers are congruent or not according to the remainder of their division by 8. 3. Tunnell's Theorem: maybe the solution of the Congruent Number Problem?

History of the Congruent Number Problem	
10th century	The Persian mathematician Al-Karaji was the first to ask the following question: how do we determine if a given positive integer is congruent?
12th-13th century	Italian mathematician Leonardo Fibonacci studies congruent numbers in his book Liber Quadratorum, making a list of the first congruent numbers and stating (but not proving) that 1 is not congruent
17th century	French mathematician Pierre de Fermat invents the method of descent and uses it to prove that 1 is not congruent. This is the first time someone proves that not all numbers are congruent.
1983	American mathematician J.B. Tunnell proves that for a number to be congruent, it has to satisfy a certain Diophantine condition. His result can be used to show that a certain number is not congruent
1990	American mathematician Paul Monsky proves a result concerning prime numbers and congruent numbers.

Learning objectives: understanding how mathematics works

1. Mathematics, like any other science, is a process in continuous evolution. Properties of numbers and of geometrical objects that we now consider "obvious" have been discovered at a specific point in time. Before the time of Pythagoras for example, people didn't know that the formula $a^2 + b^2 = c^2$ is true for any right-angled triangle.

2. Some questions, even if very easy to ask, can be extremely difficult to answer. The Congruent Number Problem is an example of such a question: the mathematical knowledge that high school provides is enough to understand what the problem is, but nobody has been able to solve it yet.

3. A problem that has not been solved yet is an "open problem". If, at some point, someone solves it, the problem ceases to be open. Generally, the solution of the problem will be formulated as a theorem, i.e. a statement that is proved to be true using a series of logical steps that follow directly one from another.

4. The Congruent Number Problem is a perfect example of open problem. Even though we know some properties of congruent numbers, we still don't have a rule to decide whether a positive integer is congruent.

5. At the time of Pythagoras, determining whether $\sqrt{2}$ was rational was an open problem. Only a century after Pythagoras, Hippasus proved that $\sqrt{2}$ is not rational.

Tutorial 1 – Pythagoras and his community

Pythagoreans' Hymn to the Rising Sun, by Fedor Andreevich Bronnikov (1869)

What is the Purpose of Tutorial 1?

- To prove Pythagoras' Theorem
- To get an idea of what Pythagoreanism was
- To understand what theorems and proofs are

Key Questions:

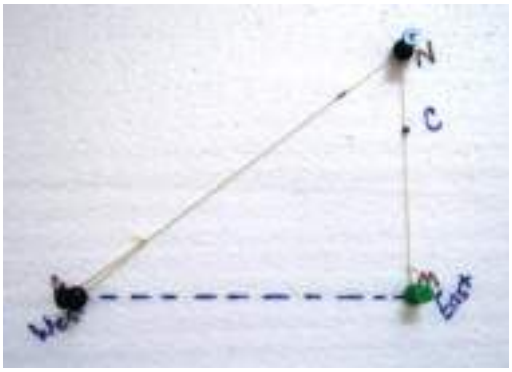
- What did Pythagoras discover about right-angled triangles?
- How do we know that what he discovered is true for ALL right-angled triangles?
- How important was maths to Pythagoras and his community?

Starter Activity:

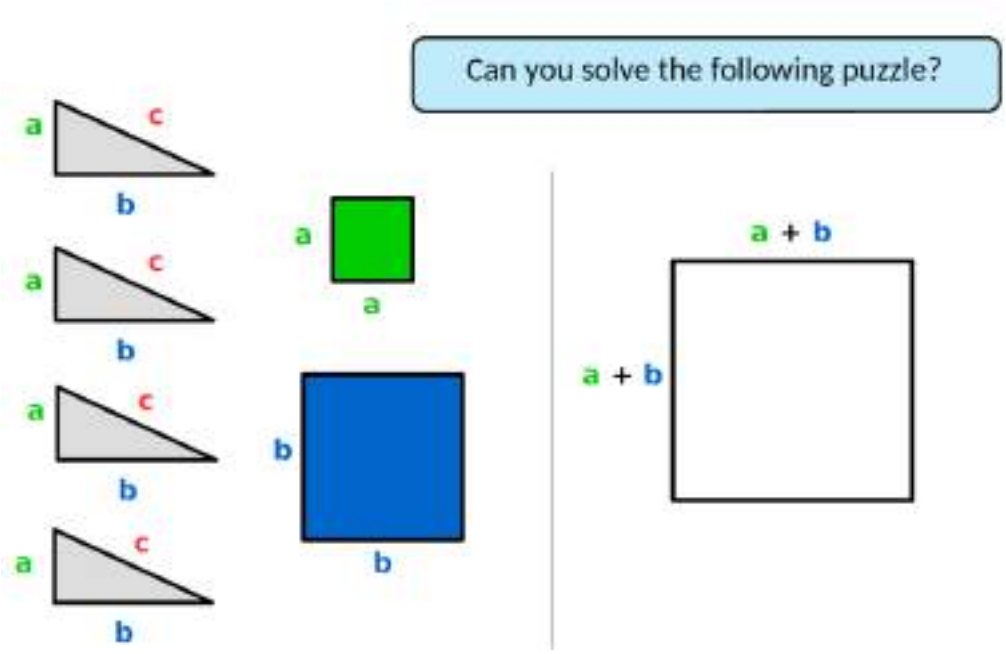
In the past, construction workers didn't have the technology we have nowadays. Still, they were able to build perfectly straight walls just using a rope. Can you figure out how they did?

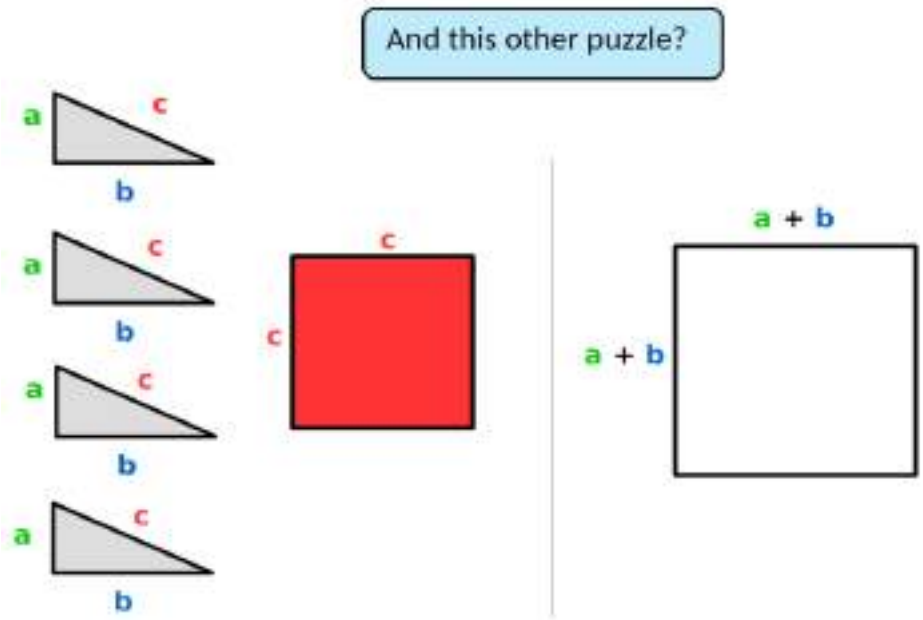
Work in pairs: you have 3 minutes to form a right-angled triangle using just a rope, a pen and a pair of scissors.

?



ACTIVITY: Using puzzles to prove Pythagoras' Theorem





Notes

Ancient Pythagorean Triples...

Plimpton 322: a 4000-year-old Babylonian clay tablet whose entries are essentially Pythagorean triples.

Homework 1: Baseline Assignment

Activity 1:

- Write a short essay (200 words or more) about Pythagoreanism. For doing this, you can use the information you were given in the first tutorial and you can also make your own research, going online. As a suggestion, you can read the Wikipedia page about Pythagoreanism.
  In your essay you can write about any aspect of Pythagoreanism that you find interesting. If you want, you can try to answer one (or more) of the following questions:
  a) What were the religious and ethical beliefs of the members of Pythagoreanism?
  b) What rituals did they practise?
  c) What did arithmetic and numbers represent for them?
  d) Were the members of Pythagoreanism interested in music?

  As explained above, you are encouraged to use online resources, but keep in mind two warnings: if you use an external resource, you should reference it, and never “copy and paste” its content (this is called plagiarism).

Activity 2:

- See today’s exercise sheet on Pythagoras’ Theorem.
  Answer clearly the questions in it, justifying every assertion you make and showing your working.

Tutorial 2 – Why was Hippasus drowned at sea?

Representation of Hippasus drowning at sea

What is the Purpose of Tutorial 2?

- To understand the difference between rational and irrational numbers
- To show that $\sqrt{2}$ is irrational
- To understand the importance of the discovery of irrational numbers

Key Questions:

- Can all numbers be expressed as fractions?
- How does the decimal expansion of a fraction look?
- Why was Hippasus drowned at sea? What was the problem with the existence of irrational numbers?

Starter Activity:

Write the decimal expansion of each of the following fractions. (You can use a calculator)

$\frac{3}{4}$ =…….. $\frac{134}{5}$ =…….. $\frac{8}{25}$ =……..

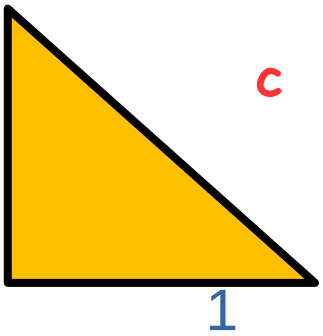
$\frac{1}{3}$ =…….. $\frac{31}{9}$ =…….. $\frac{511}{9}$ =……..

$\frac{19}{6}$ =…….. $\frac{4711}{900}$ =…….. $\frac{296}{9}$ =……..

Now work in pairs and analyze the decimal expansions above, row by row.
Identify the three possible forms that decimal expansions of fractions can assume.

ACTIVITY 1: A very unexpected consequence of Pythagoras' Theorem.

Consider the following right-angled triangle:



What is the value of **c** ?

Now write the decimal expansion of $\sqrt{2}$

$\sqrt{2}$ =……..

and compare it with the decimal expansions of the fractions above. What do you notice?

The difference in between the decimal expansion of fractions and the decimal expansion of numbers like $\sqrt{2}$ is huge!

In other terms: numbers like $\sqrt{2}$ are much more complex than fractions!

This is why they deserve their own name: we call them **irrational numbers**.

ACTIVITY 2: An algebraic proof: $\sqrt{2}$ is an irrational number!

Now we will use your knowledge about factors, prime factors and unique factorization to show that $\sqrt{2}$ is irrational!

Notes

Homework 2:

Activity 1:

- Write a short essay (200 words or more) about Hippasus of Metapontum. For doing this, you can use the information you were given in the second tutorial and you can also make your own research, going online. As a suggestion, you can read the Wikipedia page about Hippasus.
  If you want, you can try to answer one (or more) of the following questions:
  a) What do we know about his life?
  b) What discovery was he credited with?
  c) What do we know about his death?

  As explained above, you are encouraged to use online resources, but keep in mind two warnings: if you use an external resource, you should reference it, and never “copy and paste” its content (this is called plagiarism).

Activity 2:

- See today’s exercise sheet on rational and irrational numbers.
  Answer clearly the questions in it, justifying every assertion you make and showing your working.

Tutorial 3 – How come a 1000-year-old problem is still unsolved?

Al-Fakhri fi'l-jabr wa'l-muqabala (Glorious on algebra), by al-Karaji

What is the Purpose of Tutorial 3?

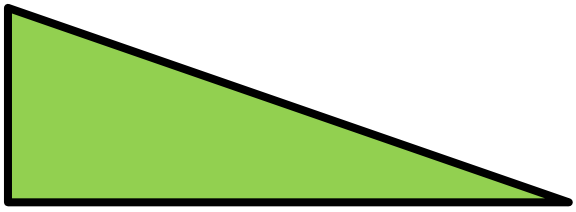
- To introduce the Congruent Number Problem
- To understand the difference between a theorem and an open problem
- To be able to give some examples of congruent numbers

Key Questions:

- What numbers occur as the area of a right-angled triangle?
- What about if we only look at right-angled triangle with rational sides?
- What is a congruent number?
- How can Pythagorean triples help us find congruent numbers?

Starter Activity:

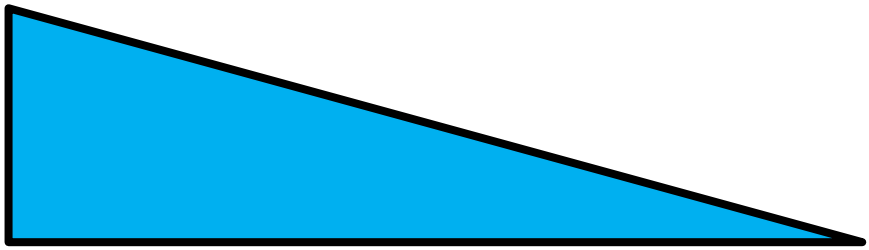
1. Easy problem: take any positive integer, for example 6. Can you find a right-angled triangle with area 6?



6

(Hint: You can start with a right-angled triangle of base 6.
What should its height be in order for it to have area 6?)

2. Now try to generalize: let n be a positive integer. Can you find a right-angled triangle with area n?



n

3. Let us make things a bit more difficult! Let n be a positive integer. Is there a rational right-angled triangle with area n?

(Hint: for example if n=6, is the triangle you constructed in question 1 a rational right-angled triangle?)

Question 3 is much more difficult to answer than question 2!

Congruent Numbers

Fibonacci was the first to give a special name to positive integers for which the answer to question 3 is YES.

A positive integer n is said to be **CONGRUENT** if there exists a rational right-angled triangle with area n.

The Congruent Number Problem

We call “Congruent Number Problem” the problem of finding which positive integers are congruent and which not.

Note: solving the “Congruent Number Problem” means knowing, <u>for each positive integer n</u>, if there is or not a rational right-angled triangle with area n.

The Congruent Number Problem is an open problem!

Mathematicians have tried to solve it for more than a thousand years but so far they have not been successful!

ACTIVITY: How to find congruent numbers?

An idea to find some (infinitely many!) congruent numbers!

Even if it is not easy to determine whether a positive integer is congruent or not, we have a very powerful instrument to find many congruent numbers!

(Hint: go back to the end of the first lesson... Look at Plimpton 322…)

Notes

Homework 3:

Activity 1:

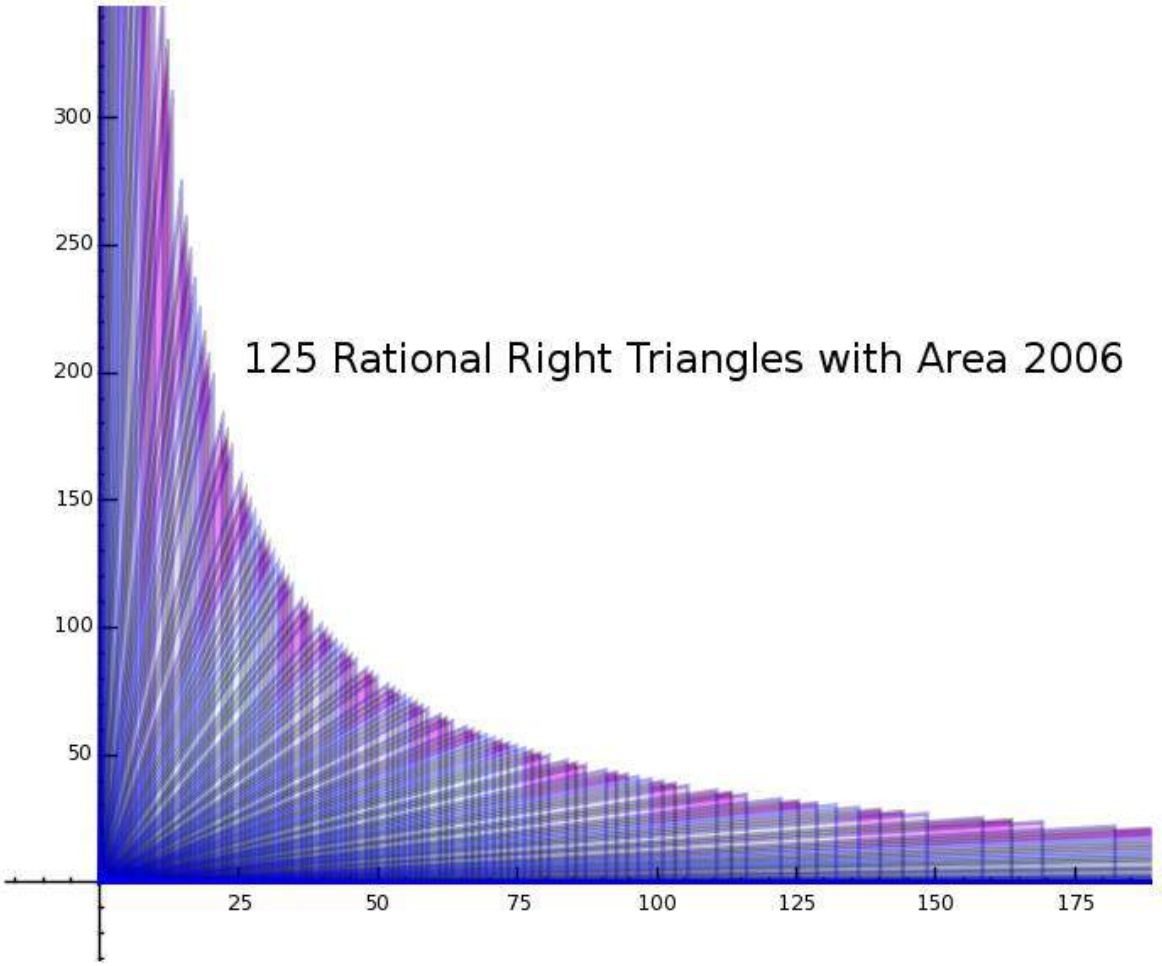
- Write a short essay (200 words or more) about Fibonacci. For doing this, you can use the information you were given in the third tutorial and you can also make your own research, going online. As a suggestion, you can read the Wikipedia page about Fibonacci.
  If you want, you can try to answer one (or more) of the following questions:
  a) What do we know about his life?
  b) What is the Fibonacci sequence?
  c) How is such sequence related to nature?

  As explained above, you are encouraged to use online resources, but keep in mind two warnings: if you use an external resource, you should reference it, and never “copy and paste” its content (this is called plagiarism).

Activity 2:

- See today’s exercise sheet on congruent numbers.
  Answer clearly the questions in it, justifying every assertion you make and showing your working.

Tutorial 4 – Are all numbers congruent?



What is the Purpose of Tutorial 4?

- To understand what a Diophantine equation is
- To familiarize with the concept of “partial solution” of an open problem
- To understand that not all numbers are congruent

Key Questions:

- Are all positive integers congruent numbers?
- What is the main characteristic of Diophantine equations?
- What does it mean for a problem to be “open”?
- What is a partial solution of an open problem?

Starter activity

1. List all the prime numbers bigger than 10 and smaller than 30.

2. For each of the numbers in question 1, say if it is congruent, if it is not congruent or if we do not know.

   To do so use the two theorems below (these theorems were proven by the American mathematician Paul Monsky in 1990):

- (Theorem of prime numbers which are congruent) If p is a prime number and the remainder of p in the division by 8 is either 5 or 7, then p is congruent.
- (Theorem of prime numbers which are not congruent) If p is a prime number and the remainder of p in the division by 8 is 3 then p is not congruent.

Fermat was the first to show that not all numbers are congruent!

In the 12th century, Leonardo Fibonacci studied congruent numbers and he wrote in his book “Liber Quadratorum” that 1 is not congruent. Unfortunately, he could not prove that statement.

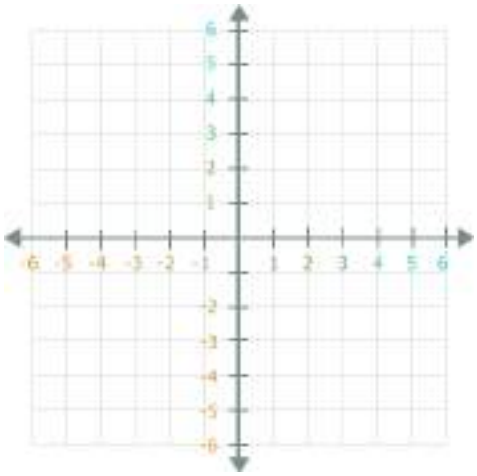
In the 17th century, after five centuries, Pierre de Fermat proved that 1 is not congruent! This result is called Fermat’s Right Triangle Theorem.

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
CVM COMMENTARIIS C. G. BACHETI V. C.
TOLOSÆ,
M. DC. LXX.

Work by Diophantus, translated from Greek into Latin by Bachet. This edition was published in 1670, and contains additions by Pierre de Fermat.

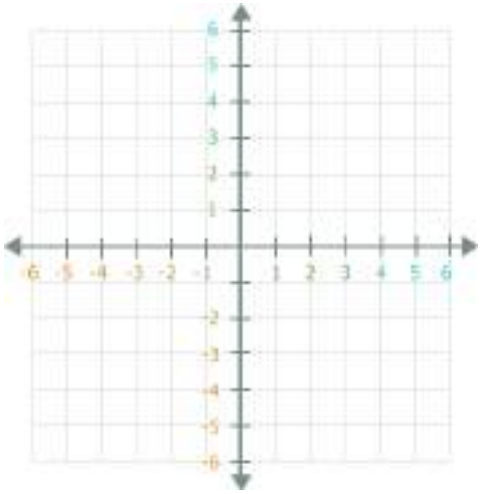
ACTIVITY: Introduction to Diophantine equations

1. Consider the simple equation $y = 3x - 2$: draw the line associated to it.



2. Now circle those points on the line whose coordinates are both integers.

3. Consider the equation $y = x^2 - 1$: draw the parabola associated to it.



4. Now circle those points on the line which have both coordinates integers.

5. Is it possible to determine all the points with integer coordinates of a graph without drawing it? Sometimes it is!

6. Consider for example the equation $y^2 = -2x^2 + 2$
   We want to determine all integer pairs (x, y) such that $y^2 = -2x^2 + 2$.

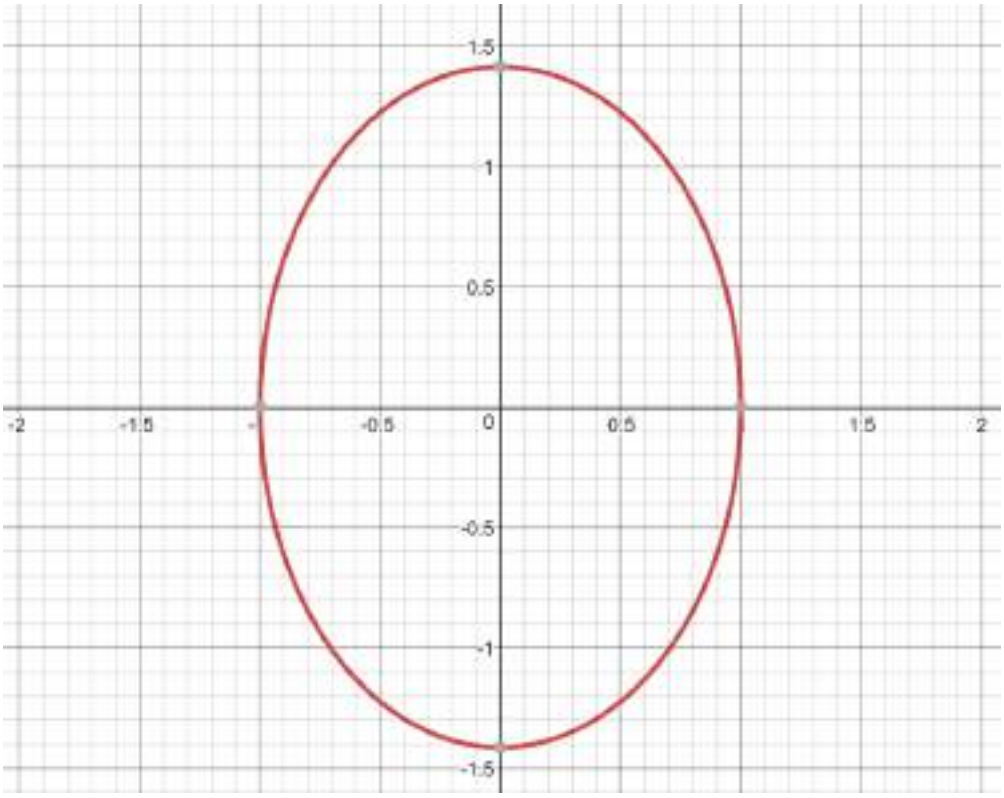
Let us start with writing our equation in the equivalent form $y^2 + 2x^2 = 2$ and checking by hand!

For example: is $(0,0)$ a point of the graph of $y^2 + 2x^2 = 2$?
is $(0,1)$ a point of the graph of $y^2 + 2x^2 = 2$?
is $(0,-1)$ a point of the graph of $y^2 + 2x^2 = 2$?
is $(1,0)$ a point of the graph of $y^2 + 2x^2 = 2$?

The same can be asked for all integer pairs
$(-1,0), (0,2), (0,-2), (2,0), (-2,0), (1,-1), (-1,1)$ etc etc

7. But we cannot check all the pairs, there are infinitely many!

Let us be smart! Start noticing that both y^2 and $2x^2$ are non-negative. Therefore…



Notes

Homework 4:

Activity 1:

- Write a short essay (200 words or more) about Pierre de Fermat. For doing this, you can use the information you were given in the fourth tutorial and you can also make your own research, going online. As a suggestion, you can read the Wikipedia page about Fermat.
  If you want, you can try to answer one (or more) of the following questions:
  a) Who was he?
  b) What was his contribution in the study of congruent numbers and why was it so important?
  c) What is he most known for?

  As explained above, you are encouraged to use online resources, but keep in mind two warnings: if you use an external resource, you should reference it, and never “copy and paste” its content (this is called plagiarism).

Activity 2:

- See today’s exercise sheet on Diophantine equations.
  Answer clearly the questions in it, justifying every assertion you make and showing your working.

Tutorial 5 – Tunnell's theorem: a criterion to find non-congruent numbers

Invent. math. 72, 323-334 (1983)

Inventiones mathematicae
© Springer-Verlag 1983

A Classical Diophantine Problem and Modular Forms of Weight 3/2

J.B. Tunnell

Department of Mathematics, Fine Hall, Box 37, Princeton University, Princeton, NJ 08540, USA

to S. Chowla

Introduction

It is a classical Diophantine problem to determine which integers are the area of some right triangle with rational sides. The main result of this paper is the following.

Theorem. *Let formal power series in the variable q be given by $g = q\prod_1^\infty (1-q^{8n})(1-q^{16n})$ and, for each positive integer i, $\theta_i = \sum_{-\infty}^{\infty} q^{i n^2}$. Set $g\theta_2 = \sum_1^\infty a(n) q^n$ and $g\theta_4 = \sum_1^\infty b(n) q^n$.*

(a) *If $a(n) \neq 0$, then n is not the area of any right triangle with rational sides.*

(b) *If $b(n) \neq 0$, then $2n$ is not the area of any right triangle with rational sides.*

The power series $g\theta_2$ and $g\theta_4$ are the q-expansions of certain modular forms of weight 3/2. It would follow from some current conjectures in the theory of elliptic curves that the converses of statements (a) and (b) are true for square-free positive integers n (see Sect. 3).

In this introduction we will briefly recall the history of the problem and the connections to elliptic curves, as well as giving a description of the methods of the paper.

Let $\mathscr{C}$ be the set of areas of right triangles with rational sides. This is a subset of $(\mathbf{Q}^*)^+$, and consideration of similar triangles shows that it is a union of cosets of $(\mathbf{Q}^*)^2$. Classically, an integer in $\mathscr{C}$ was called a congruent number, and Dickson [9, Chap. XVI] traces the question of whether a given number is congruent back to Arab manuscripts and the Greeks prior to that. The positive integers not in $\mathscr{C}$ are called noncongruent numbers, a terminology we will use unless confusion with ideal theoretic congruence might result.

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The first page of Tunnell's article in the journal "Inventiones mathematicae"

What is the Purpose of Tutorial 5?

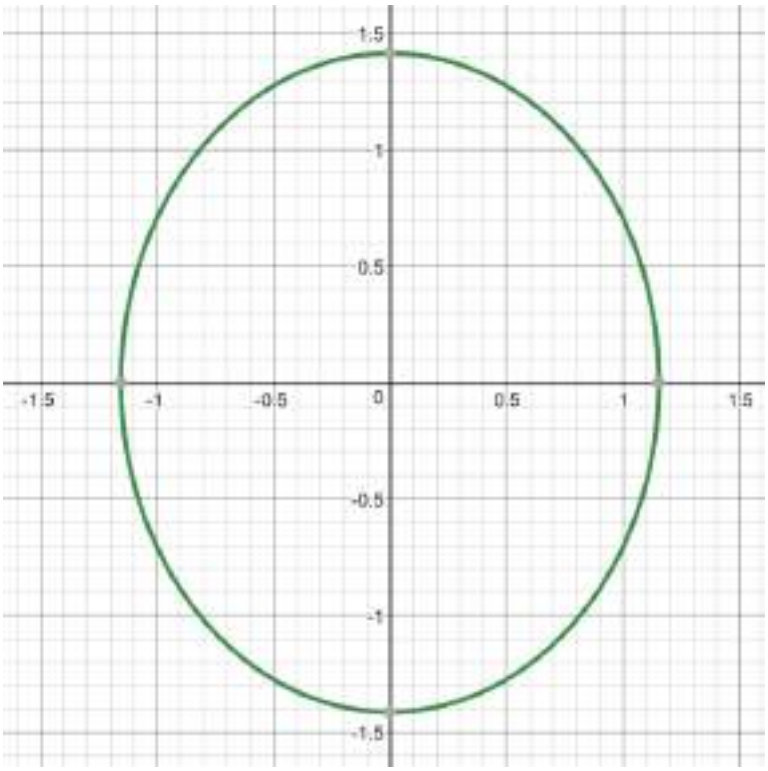
- To understand how to find integer solutions to polynomial equations of the type $x^2 + 3y^2 + 2z^2 = 5$
- To introduce Tunnell's theorem
- To use Tunnell's theorem to show that 1 is not congruent

Key Questions:

- What is the most important contribution of the mathematician J.B. Tunnell?
- What does an equation with 3 variables represent?

Starter activity

In the last tutorial we have introduced the notion of Diophantine equation. To be sure you remember what you have learnt, find all the pairs of integers (x, y) such that $3x^2 + 2y^2 = 4$.



Let us add one more variable!

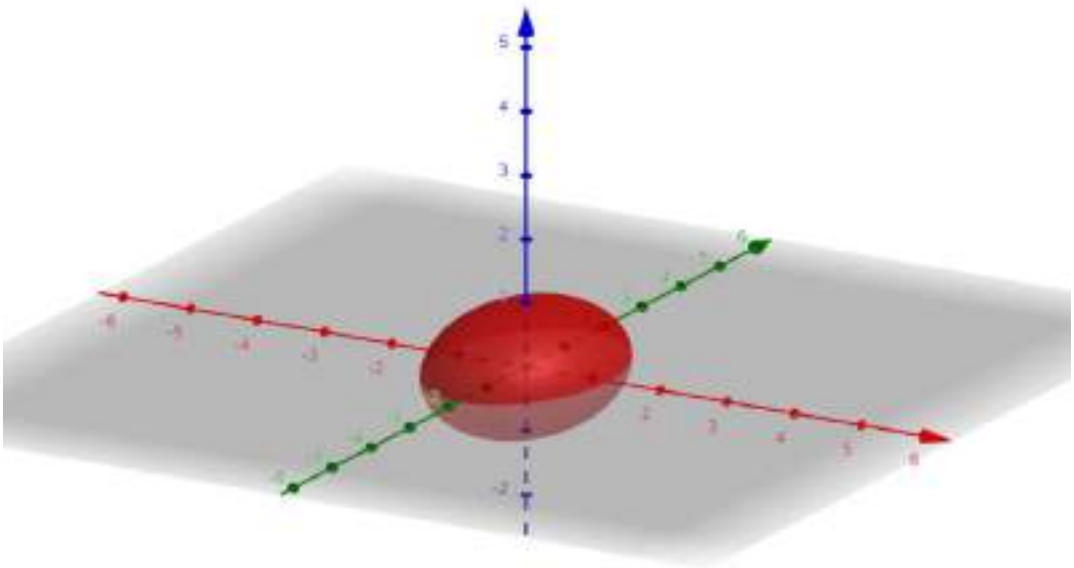
ACTIVITY:
Instead of considering equations in two variables x and y, we can work with three variables: x,y and z.

Let us analyze the following equation: $2x^2 + y^2 + 3z^2 = 3$.

As before, we are interested in finding all triples of integers (x, y, z) such that $2x^2 + y^2 + 3z^2 = 3$.

Note that a triple (x, y, z) represents a point in the three-dimensional space, while a pair (x, y) represents a point in the plane.
This said, the method to find all triples of integers such that $2x^2 + y^2 + 3z^2 = 3$ is the same as the method for equations with two variables.

Try!



Tunnell's Theorem

Another partial solution to the Congruent number problem is Tunnell's Theorem:

If n is an odd congruent number, then the number of points with integer coordinates in the graph of $2x^2 + y^2 + 32z^2 = n$ is half the number of points with integer coordinated in the graph of $2x^2 + y^2 + 8z^2 = n$.

We can use Tunnell's Theorem when we want to show that a certain odd number is not congruent.

APPLICATION OF TUNNELL'S THEOREM:
Let us try with 1! Fermat proved that 1 is not congruent. Let us prove this fact by ourselves using Tunnell's Theorem!

1. Find all the points with integer coordinates in the graph of $2x^2 + y^2 + 32z^2 = 1$. How many points did you find?

2. Find all the points with integer coordinates in the graph of $2x^2 + y^2 + 8z^2 = 1$. How many points did you find?

3. Is the number you found answering question 1 half the number you found answering question 2?

Notes

Homework 5 - Final Assignment

Activity 1:

Write an essay (around 1000 words) or make a poster presentation on the history of the Congruent Number Problem and of the mathematicians that worked on it. Explain what the problem consists in and when it was first asked. Describe then the various partial results we have encountered during the course, with particular attention to Fermat's right triangle theorem and Tunnell's theorem. Make sure to clarify the difference between an open problem (such as the Congruent Number Problem itself) and partial results (such as Fermat's right triangle theorem, Tunnell's theorem, …).
As for the previous homework assignment, you are encouraged to use online resources, but keep in mind two warnings: if you use an external resource, you should reference it, and never "copy and paste" its content (this is called plagiarism).

Activity 2:

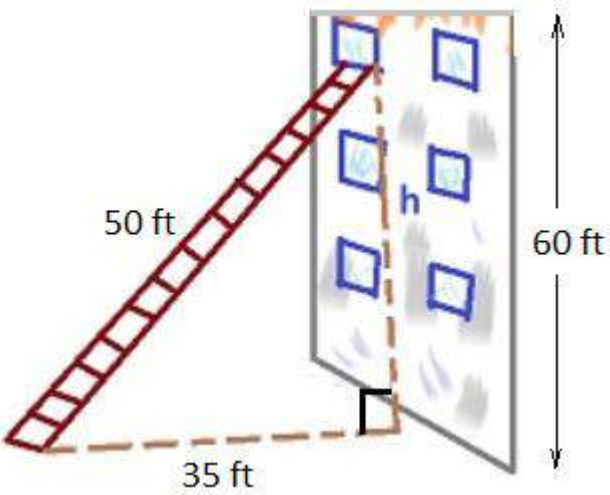
Answer clearly the following questions, justifying every assertion you make and showing your working.

1. A) Give an example of a rational number which is not an integer.

   B) Give three examples of irrational numbers.

2. A water park wants to add a zipline into a pool. If the platform at the top of the zipline is 25 feet tall, and the pool is 40 feet long, what is the maximum length needed for the zipline?

3. A 50 feet ladder is placed 35 feet from a wall. The distance from the ground straight up to the top of the wall is 60 feet. Check whether the ladder reaches the top of the wall.



4. Give three examples of congruent numbers. For each number exhibit a rational right triangle with area such number.

5. Use the theorem on prime and congruent numbers to find at least 5 prime congruent numbers bigger than 100.

(Hint: you can find online the list of prime numbers from 100 to 200 for example)

6. Use the theorem on prime and non-congruent numbers to find at least 3 prime non-congruent numbers bigger than 100.

(Hint: you can find online the list of prime numbers from 100 to 200 for example)

7. Use Tunnell's theorem to show that 3 is not congruent.

(Hint: count the number of integer triples (x,y,z) such that $2x^2 + y^2 + 32z^2 = 3$.
Do the same with the equation $2x^2 + y^2 + 8z^2 = 3$ and then compare the numbers you obtain...)

Tutorial 6 – Feedback tutorial

What is the Purpose of Tutorial 6?

- To receive feedback on final assignments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.
- To reflect on the programme including what was enjoyed and what was challenging.

Final assignment feedback

What I did well…	What I could have improved on…
•	•
•	•
•	•

My target for future work is…

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?
• • •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

Appendix 1 – Referencing correctly

When you get to university, you will need to include references in the assignments that you write, so we would like you to start getting into the habit of referencing in your Brilliant Club assignment. This is really important, because it will help you to avoid plagiarism. Plagiarism is when you take someone else's work or ideas and pass them off as your own. Whether plagiarism is deliberate or accidental, the consequences can be severe. In order to avoid losing marks in your final assignment, or even failing, you must be careful to reference your sources correctly.

What is a reference?

A reference is just a note in your assignment which says if you have referred to or been influenced by another source such as book, website or article. For example, if you use the internet to research a particular subject, and you want to include a specific piece of information from this website, you will need to reference it.

Why should I reference?

Referencing is important in your work for the following reasons:

- It gives credit to the authors of any sources you have referred to or been influenced by.
- It supports the arguments you make in your assignments.
- It demonstrates the variety of sources you have used.
- It helps to prevent you losing marks, or failing, due to plagiarism.

When should I use a reference?

You should use a reference when you:

- Quote directly from another source.
- Summarise or rephrase another piece of work.
- Include a specific statistic or fact from a source.

How do I reference?

There are a number of different ways of referencing, and these often vary depending on what subject you are studying. The most important to thing is to be consistent. This means that you need to stick to the same system throughout your whole assignment. Here is a basic system of referencing that you can use, which consists of the following two parts:

1. **A marker in your assignment**: After you have used a reference in your assignment (you have read something and included it in your work as a quote, or re-written it your own words) you should mark this is in your text with a number, e.g. [1]. The next time you use a reference you should use the next number, e.g. [2].
2. **Bibliography**: This is just a list of the references you have used in your assignment. In the bibliography, you list your references by the numbers you have used, and include as much information as you have about the reference. The list below gives what should be included for different sources.

    a. **Websites** – Author (if possible), title of the web page, website address, [date you accessed it, in square brackets].

    E.g. Dan Snow, 'How did so many soldiers survive the trenches?', http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [11 July 2014].

    b. **Books** – Author, date published, title of book (in italics), pages where the information came from.

    E.g. S. Dubner and S. Levitt, (2006) *Freakonomics*, 7-9.

    c. **Articles** – Author, 'title of the article' (with quotation marks), *where the article comes from* (newspaper, journal etc.), date of the article.

    E.g. Maev Kennedy, 'The lights to go out across the UK to mark First World War's centenary', Guardian, 10 July 2014.

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Mathematical Mysteries: What are right-angled triangles hiding from you?

Key Stage 4 Programme

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