Integrability - I can’t even say it!

Key Stage 4 Programme

Pupil Name	
Handbook Designed by	Dr. Hills

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1			
2			
3			
4			
5			
6 (Feedback)			

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1		
Tutorial 2		
Tutorial 3		
Tutorial 4		
Tutorial 5		

Assignment Submission – Lateness and Plagiarism

Lateness	
Submission after midnight on 19 July	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

The Brilliant Club KS4 Programme – Pupil Feedback Report

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-level
2:1	60-69	Performing to a good standard at AS-level
2:2	50-59	Performing to an excellent standard at GCSE
3rd	40-49	Performing to a good standard at GCSE
Working towards a pass	0-39	Performing below a good standard at GCSE
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

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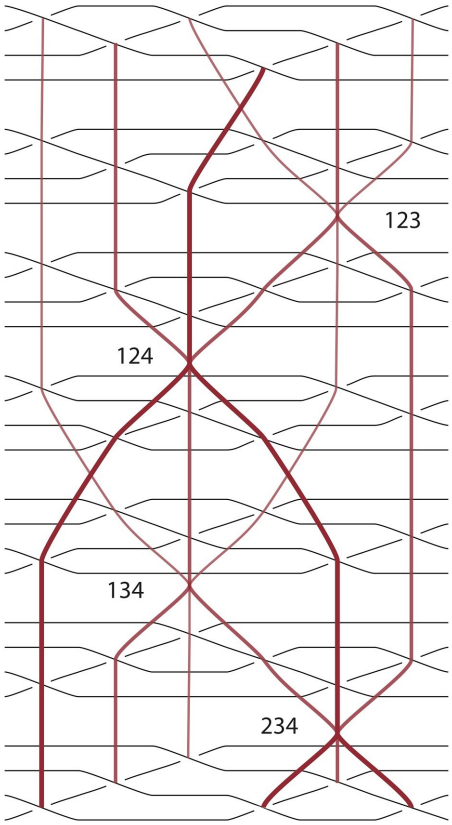
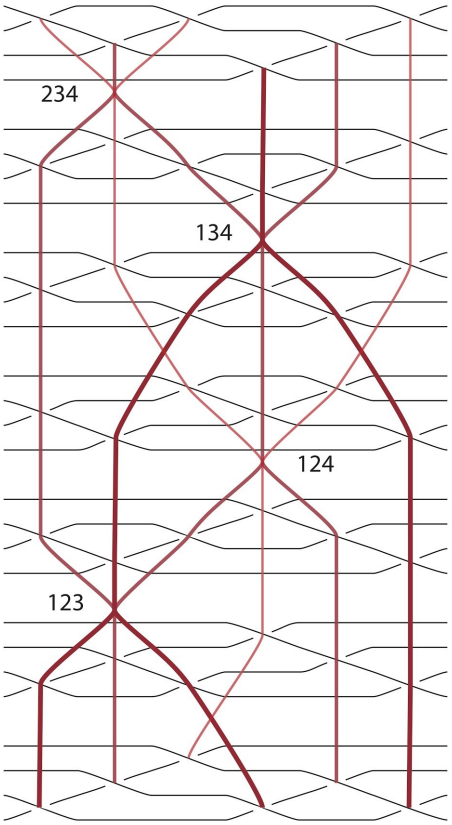
Course Rationale

This course is intended to expose students to different concepts within the Mathematical (and Physical) research area of **Integrability.** Students will learn about the early days of Integrability, when Johannes Kepler studied the **motion of the planets** and when mathematicians were interested in the dynamics underpinning bizarre **spinning tops**. Within wider science, Integrability has played an important role. Integrable systems have been used as toy models many times, which allows us to gain a deeper initial understanding of physical phenomena. For example, its features appear in the world of **Nonlinear Optics.** Scientists, and mathematicians, were therefore able to describe the characteristics of the theory much more concretely.

Students will learn some of the key features of an **Integrable System**, appreciate the **importance of energy conservation** and the role it plays within Integrability and wider science. In order to do this, students will learn how to find the derivative of several different types of function, which is usually encountered at A-level. We will work through this course handbook together, completing the tasks and discuss wider research.

By the end of the course, students will hopefully have a better understanding of the fact that many, if not all, scientific subjects are connected. And that there are many curious objects in the world of scientific research, which still need to be studied in great detail!





The diagram to the left illustrates the

Tetrahedron Equation.

It is the equation that describes the necessary condition for integrability within 3-dimensional systems. If a system is Integrable, both sides of the diagram are equal.

Each vertical coloured strand represents a particle trajectory and the numbers denote which particles are interacting at that point.

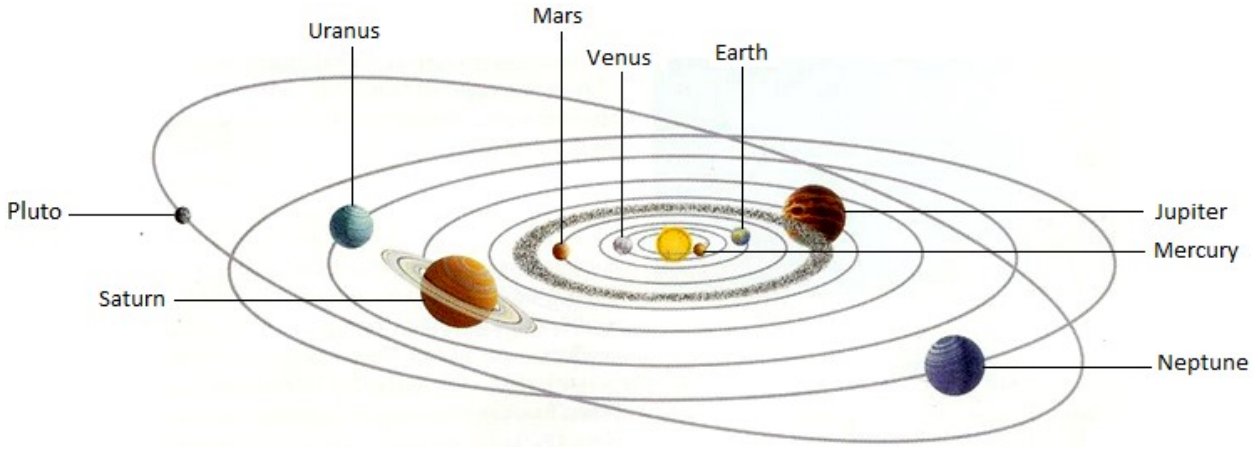
Mark Scheme Table

	Skills	1st (70-100)	2:1 (60-69)	2:2 (50-59)
Subject knowledge	Knowledge and Understanding	• All content included is relevant to the general topic and to the specific question/title • Good understanding of all the relevant topics. • Scientific terms are defined and used accurately throughout • Clear justification on how the content included is related to the specific issues that are the focus of the assignment	• Most of the content included is relevant to the general topic and to the specific question/title • Good understanding of most the relevant topics • Scientific terms are used accurately but not always clearly defined. • Adequate justification on how the content included is related to the specific issues that are the focus of the assignment	• Some of the content included is relevant to the general topic and to the specific question/title • Good understanding on some of the relevant topics but occasional confusion on others. • Scientific terms are used mostly accurately with occasional confusion and often not defined. • Some justification on how the content included is related to the specific issues that are the focus of the assignment
Critical Thinking	Research and Evidence	• Inclusion of rich sources of research findings, data, quotations or other sourced material as evidence for the claims/ ideas • Use evidence/ calculations to support claims/assertions/ ideas, consistently clearly and convincingly • Evidence of further reading beyond materials provided which were used in an appropriate context	• Inclusion of adequate sources of research findings, data, quotations or other sourced material as evidence for the claims/ ideas • Use evidence/calculations to support claims/assertions/ideas, mostly clearly and convincingly • Evidence of further reading beyond materials provided	• Inclusion of some sources of research findings, data, quotations or other sourced material as evidence for the claims/ ideas • Use evidence/ calculations to support claims/assertions/ ideas, at times clearly and convincingly • Limited evidence of further reading beyond materials provided

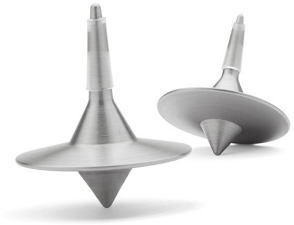
Critical Thinking	Developing an Argument	• A point of view or position in relation to the title or question is consistently clear • Argument/proof exceptionally well-developed and well-justified • Makes links effectively between subjects that have not previously been associated • Uses concepts from the tutorials in an unfamiliar context, and does so accurately and confidently	• A point of view or position in relation to the title or question is adequately clear • Argument/proof clear and well-developed and position justified • Some evidence of linking subjects that have not previously been associated • Use some concepts from the tutorials in an unfamiliar context, but not always accurate	• A point of view or position in relation to the title or question is somewhat clear • Argument/proof clear but not well-developed • Limited evidence of linking subjects that have not previously been associated • Limited use of concepts from the tutorials in other contexts
	Critical Evaluation	• Moved beyond description to an assessment of the value or significance of what is described • Evaluative points are consistently explicit/ systematic/reasoned/ justified	• Mostly description but some assessment of the value or significance of what is described • Evaluative points are mostly explicit/ systematic/reasoned/ justified	• Only description with minimal assessment of the value or significance of what is described • Evaluative points are at times explicit/ systematic/reasoned/ justified

Written Communication	Structure and Presentation	• Ideas are presented in paragraphs and arranged in a logical structure that is appropriate for the assignment • The introduction clearly outlines how the essay/ report will deal with the subject matter • The conclusion summarises all the main points clearly and concisely • All calculations, formulae and methods are clearly structured, clear to follow and correct • All sources are referenced correctly in an agreed format	• Ideas are presented in paragraphs and arranged in a structure that is mostly appropriate for the assignment • The introduction adequately describes how the essay/report will deal with the subject matter • The conclusion summarises most of the main points clearly • Calculations, formulae and methods are mostly structured, clear to follow and correct • Most sources are referenced correctly in an agreed format	• Ideas are presented in paragraphs and arranged in a structure • The introduction mentions how the essay/report will deal with the subject matter • The conclusion summarises some of the main points clearly • Calculations, formulae and methods are not always structured, clear to follow and correct. • Some sources are referenced correctly in the agreed format with occasional errors
	Language and Style	• No spelling, grammar or punctuation errors • Units and significant figures are presented accurately throughout • Writing style consistently clear, appropriate for scientific documents and easy to follow • Accurate and consistent use of technical language and vocabulary	• Minimal spelling, grammar or punctuation errors • Units and significant figures are presented accurately throughout • Writing style mostly clear, appropriate for scientific documents and easy to follow • Some attempts of using technical language and vocabulary, but not always accurate	• Some spelling, grammar or punctuation errors • Units and significant figures are presented accurately throughout with occasional errors • Writing style moderately clear, appropriate for scientific documents and easy to follow • Use of simple language and vocabulary effectively but struggles to use technical language

Tutorial 1 – From Planets to Spinning Tops



What is the Purpose of Tutorial 1?



- To detail a brief history of Integrability (and how to say it).
- To address what Integrability is in a nutshell.
- And to answer the question: "Why is it important to study Integrability in the Mathematical World?"

A Brief History of Integrability:

In the 17th century Johannes Kepler proposed three laws that govern **Planetary Motion**:

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

At the time, these laws were a huge leap forward in scientific thinking, as a lot of people still believed that the Earth was at the centre of our Solar System. Powerful Clergymen worried that new Scientific theories concerning the stars and other objects in the Universe (also known as Celestial Bodies) would lead to people questioning their faith in a God or Gods and possibly deserting the Church. However, despite Science's impact on the societies and regimes of the time, Johannes Kepler's three laws are actually one of the earliest examples of an **Integrable System**; although, Kepler was never aware of this.

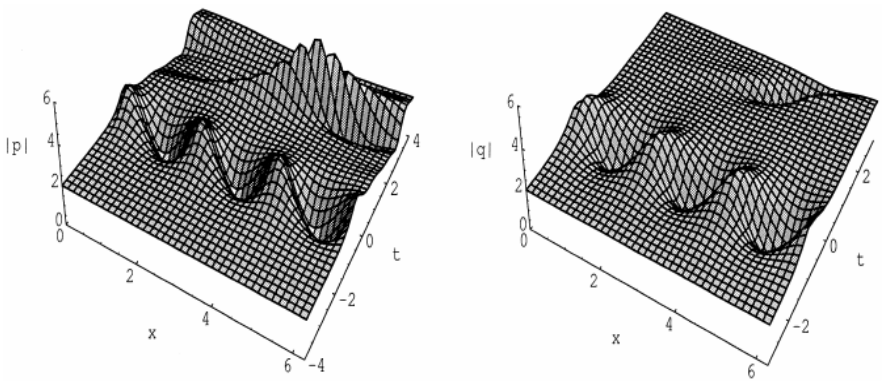
Johannes Kepler (1571—1630)

An incredibly famous German mathematician, astronomer and astrologer that played in a crucial role in the Scientific Revolution of the 17th century.

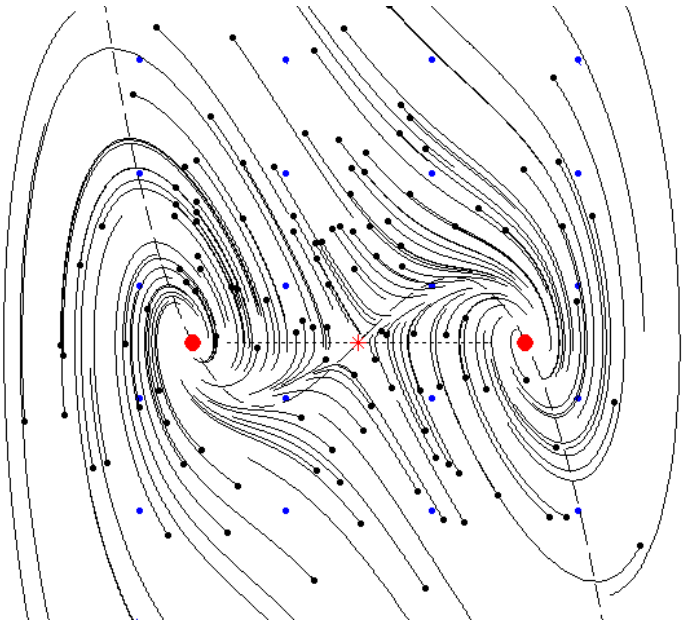
So, what is an Integrable System?

Since Integrable Systems were discovered many, many years ago the Academics that have studied them in excruciating detail have not been able to settle on a specific definition! This is often quite common in the world of scientific research! Mathematicians and Physicists often like to try to be funny when naming theorems, equations and general results. A good example of this is an important theorem in algebraic topology called the "Hairy Ball Theorem". Instead of a snazzy name, or universal definition, Academics settled on the fact that an Integrable System is any such system that exhibits a number of specific features. These may include, but are not limited to, the existence of :

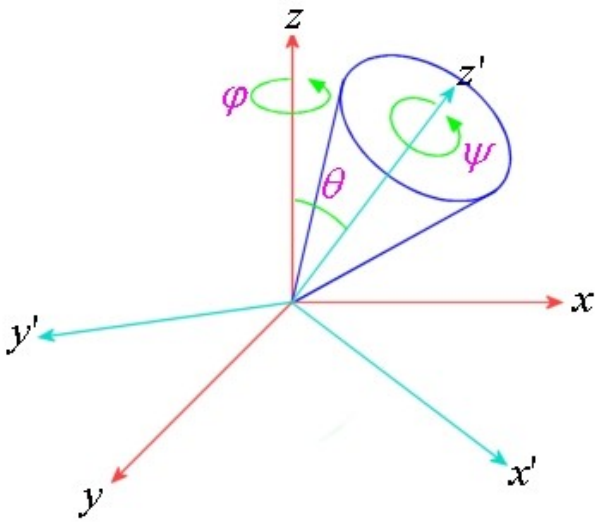
- **Bäcklund transformations,**
- **Soliton solutions,**
- **An infinite number of conserved quantities.**



All of the above are very interesting in their own right.
The image above illustrates a Bäcklund transformation and the relation between its parameters over time. For our purposes, we will focus on the idea that the **existence of an infinite number of conserved quantities** is sufficient to **guarantee** that the **system is Integrable**. In essence, this means that there are enough relations between the variables (or parameters) of your system so that you can solve it exactly, which is quite unusual in the Physical world! Normally, there is always an element of uncertainty that prevents you from being able to obtain an exact solution describing how your system evolves over time. Since we are thinking about the variables within a system and relations between them, do you notice anything striking or peculiar about the above image of a Bäcklund transformation? If you would like to know more about Integrable systems, please search: "*solitons*" in Wikipedia where you will find an animation that describes their motion.



From now on, it will be useful to always think about: the parameters, the way in which they might interact, as well as the features of the system, and whether there exist enough quantities to guarantee **Integrability.** We know that Kepler's laws provide enough information to determine a planet's orbit exactly over time, but what other systems were discovered to be Integrable?



Spinning Tops

In the 18th century another very famous mathematician, Leonhard Euler, discovered a very special spinning top, aptly named **the Euler Top**.

In the world of mathematics and physics you will often hear the word **symmetry**. This is an incredibly important concept, because it actually means that the system you are dealing with possesses some very nice relations. These relations could help you to identify **conserved quantities**, which might help us to deduce that a **system is Integrable**.

The Euler Top is a very idealised system, by which we mean that it would not really exist in the real world that we live in. It is a spinning top that is moving without the influence of any other force (a free top) and moves without any external torque (rotational force). As we know, a spinning top spins on one point, which we will call the fixed point. The fixed point of the Euler top must also coincide with the top's centre of gravity.

Now, take a moment to study the above picture that represents a spinning top and its parameters, what do you think they are?

Leonhard Euler (1707 - 1783)

Another very famous Mathematician that uncovered/ discovered many fundamental mathematical relations, which are still used heavily by researchers today.

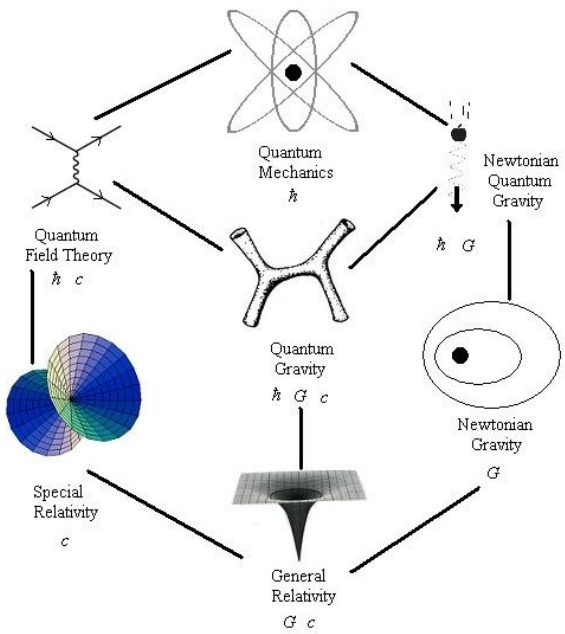
In the years that followed Euler's work several other mathematicians studied spinning tops, namely, Joseph-Louis Lagrange and Sofia Kowalevsky. In particular, they considered tops that possess **special symmetry**. The Lagrange top has an axis of symmetry where the fixed point also lies. They also consider the spinning top's resistance to angular acceleration, called **moments of inertia**, (remember that Euler did not consider any rotational force). Again, we can already tell that this is a very specialised top. The mathematics underpinning the Lagrange top is more complicated because it considers force and uses the relations between moments of inertia to solve the system exactly. Later work by Sofia Kowalevsky is yet more complex, because her top has a centre of mass that does not lie on the axis of symmetry, but perpendicular to it. As you might imagine, this makes the motion of the top more intricate as well as the maths! It is worth mentioning a little bit about Sofia Kowalevsky, she was a very interesting character.

Sofia Kowalevsky (1850 -1891)

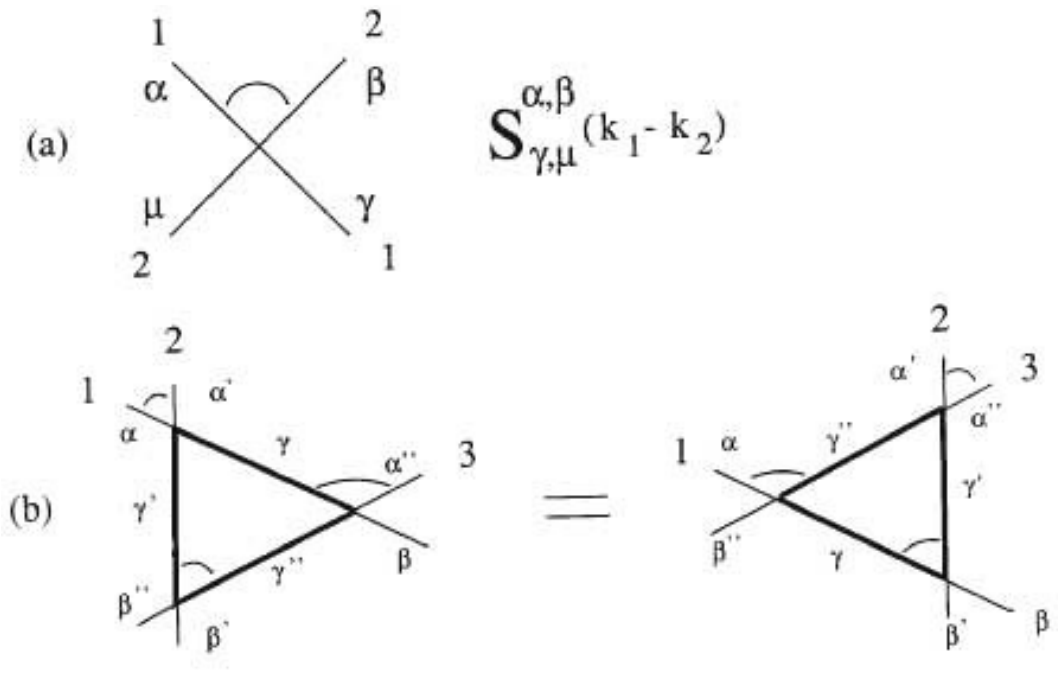
A less well known Mathematician and remarkable woman. She was a mathematician at a time when women were thought to be incapable of abstract thought! During the 1870s women were not allowed to attend University in Germany! However, Sofia remained resilient and fought for equality; she studied maths privately with willing University Professors. Eventually, in 1874, she became the first woman in modern Europe to be awarded a doctorate in mathematics from the University of Göttingen.

At this moment, we have discussed some of the earliest work within the field of Integrable Systems, as well as considering some of their typical features. Initially, it might seem strange that Scientists and Mathematicians wanted to study bizarre spinning tops, but their findings were very important. Their results pointed towards a deeper understanding of dynamical systems and ultimately Integrable Systems. By studying the features of each different spinning top - all with different characteristics - other Scholars were able to use similar techniques to make advances in their own work; some were inspired by this work and asked similar questions in other areas of Science.

Modern Day Integrability



There is still much to discover and understand about Integrable Systems. Over the last 50 years Integrability has enjoyed a resurgence in interest and curiosity. During this time, many new results have been found; in particular, several systems were connected to the real world! Previously, we mentioned the existence of **solitonic solutions** as a key marker of Integrability and they are studied heavily within the systems under many fascinating conditions, but importantly they were found to relate to the world of **Optics and Optical Fibres.** When Integrable Systems contain boundaries and/or defects, which can be thought of as walls and cracks in glass respectively, their features appear to relate to other boundary problems in **String Theory** (a controversial active research area in maths and physics). Even though some systems are not physical and might not have significance in the real world, they contain beautiful mathematical formulae. Thus **providing links between** different research areas, such as: **Abstract Algebra, Geometry and Mathematical Physics.**



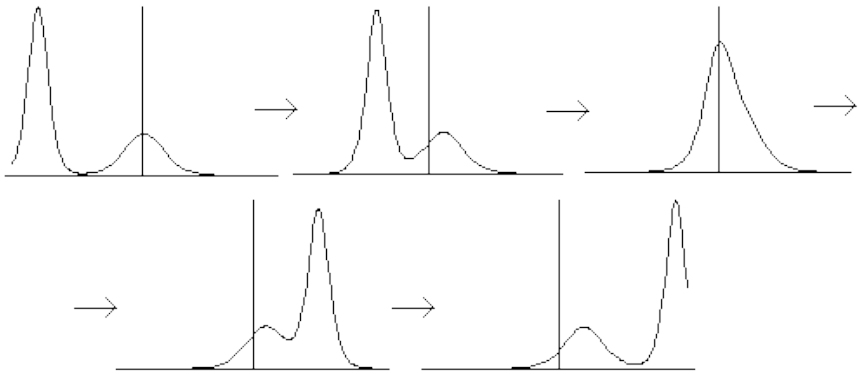
Why Study Integrable Systems?

Over the course of their history **Integrable Systems** were often used as **toy models** that allow us to gain **insight** into deeper and complex problems. A good example of this is the **Skyrme model** - which Tony Skyrme worked on for many years even though other Academics did not take his work seriously until many years later - it was proposed as a model for the Nucleon. The model allows you to make quite accurate predictions about the Nucleon and its properties. It also has application within String Theory and other areas of physics.

Tony H. R. Skyrme

(1922 - 1987)

As you can begin to see, these models and their study have an impact on other wider areas of research outside mathematics. We can't delve too deep into the different branches of Integrability but we will use the notion of conserved quantities and relations between different parameters to see how this all begins to fit together!



Homework 1 - Baseline Test

Do not worry if you do not know the answer to all of these questions. The purpose of this homework is to help ensure that our future tutorials run smoothly.

1. Name as many different types of energy, an example of that particular type of energy. (If you know the definition of that energy, for example: its defining equation, write that down too.)

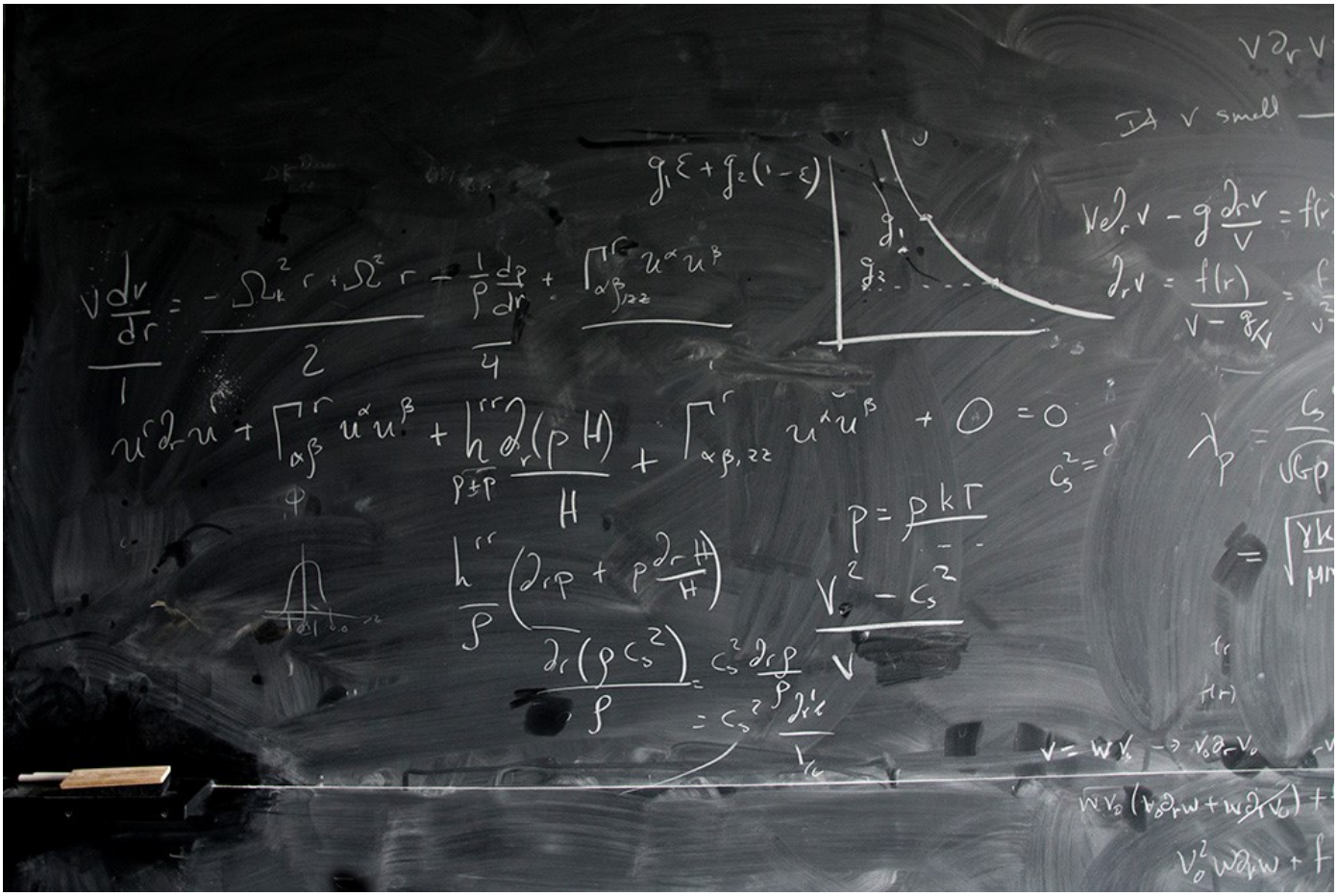
2. On x and y axes that range from –10 to +10 draw the following lines defined by:

$y = -2x + 2$,

$y = 0.5x - 5$.

3. **F** is proportional to **M.** When **M** is 450, **F** has the value 1350. How are **M** and **F** proportional? Write an equation to express this relation.

4. What is the definition of an ellipse?



Tutorial 2 – Maths in Physics and Physics in Maths

What is the Purpose of Tutorial 2?

- To appreciate that all Scientific subjects are related.
- We will examine and illustrate Kepler's three laws, so that we can see how all aspects of the system are determined, and how this hints towards conserved quantities.
- To introduce important quantities that are fundamental within Integrability, such as Conserved Total Energy and Angular Momentum.

Recalling Kepler's Laws

Recall Kepler's Three Laws of Planetary Motion:

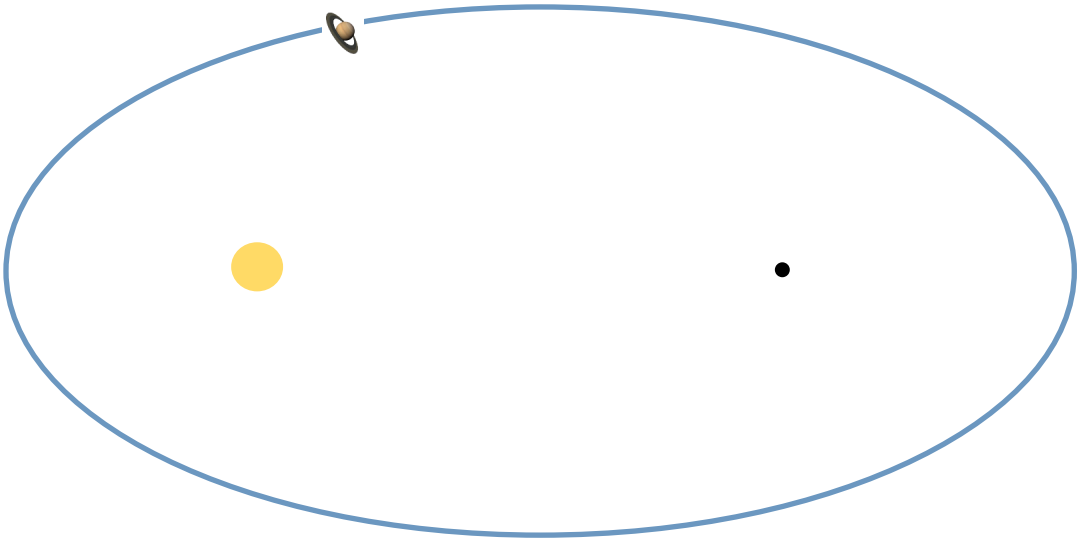
1.

2.

3.

Mathematical Formulation of Kepler's Laws

1.



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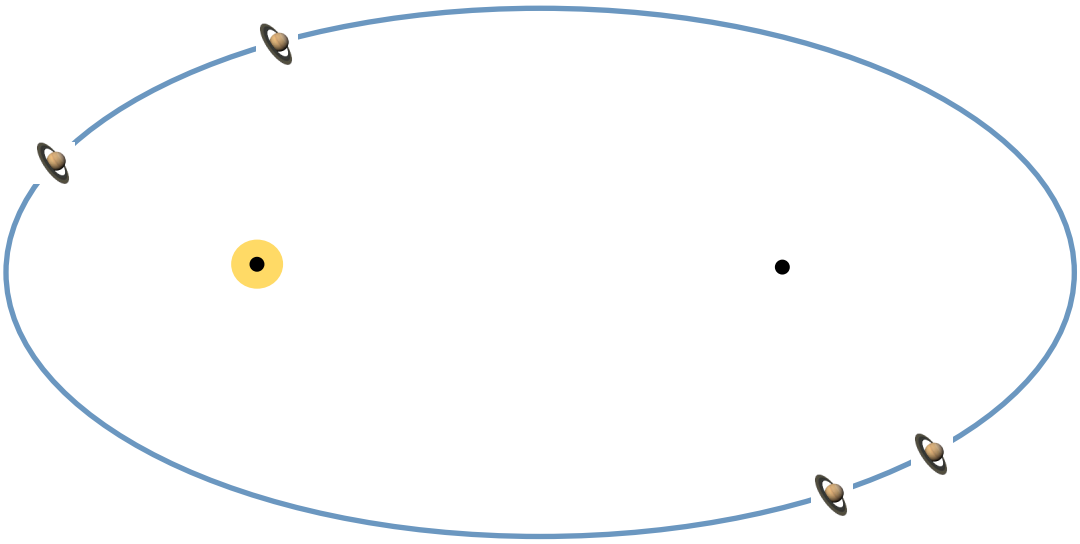
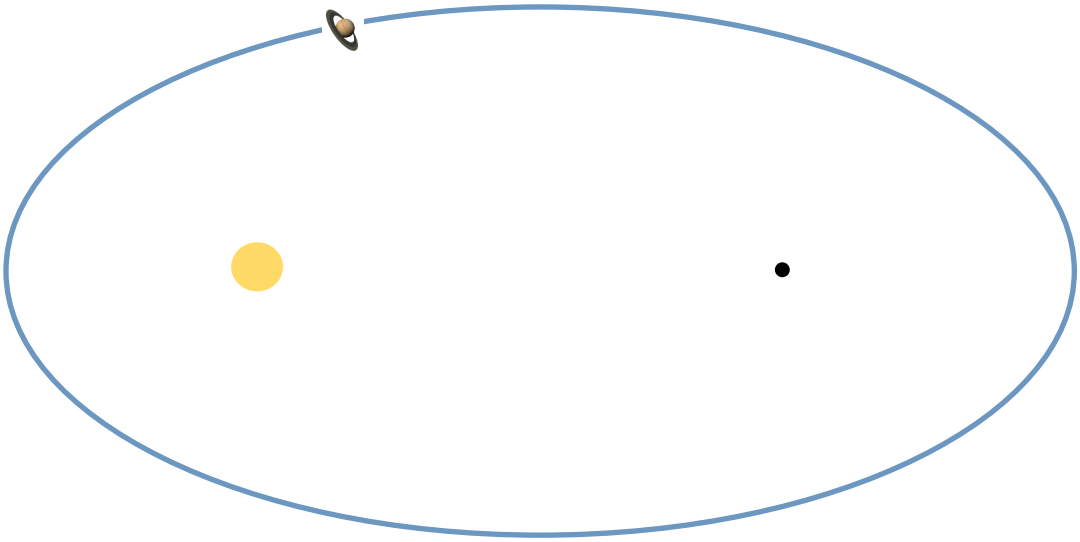


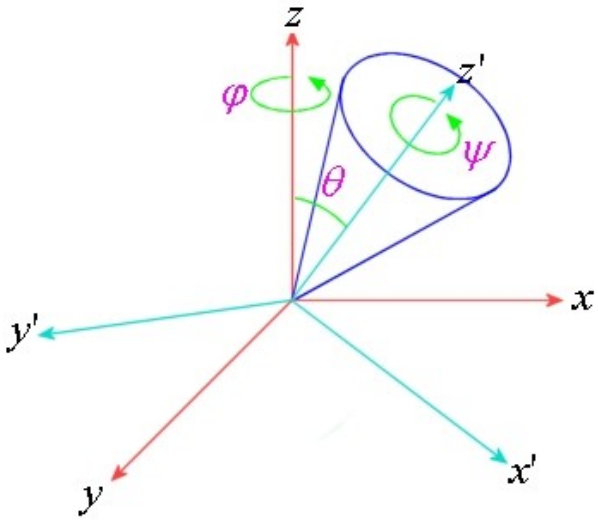
Mathematical Formulation of Kepler's Laws Continued.

3.



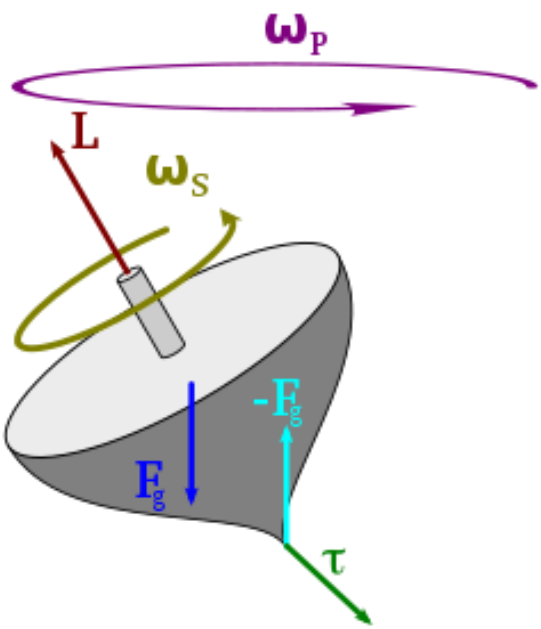
Spinning Tops Revisited

We now want to look at other objects and consider their dynamics. Let us return to our old favourite, The Spinning Top.



The image to the right depicts a spinning top moving in its own three-dimensional frame - with the axes: x', y', z', within a general spatial frame labelled by axes: x, y, z.

Below is another illustration of a Spinning Top, but now the spatial axes and its own frame's axes have been removed. Instead, several important quantities are labelled, but what are they? List them in the space below, as we run through the tutorial. We will need to use them again in the future.



Homework 2 - You spin me right round!

For this homework, think about the different quantities belonging to any spinning top that we have discussed during the tutorial. Write a short essay (max. 250 words) addressing how these quantities might change during a normal spinning top's motion and why is this not an Integrable System? You might find it helpful to draw some pictures to illustrate your ideas.

Homework 2 - You spin me right round! (continued)

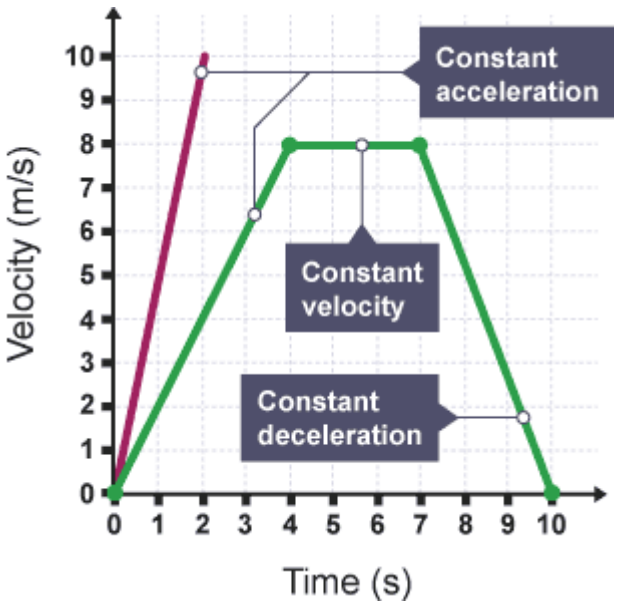
Tutorial 3 – Proving Conservation Part I

What is the Purpose of Tutorial 3?

- To define a conserved quantity mathematically.
- Introduce the mathematical operation of **Differentiation.**

Defining a Conserved Quantity

In our previous tutorials we have mentioned conserved quantities a lot, and we know that they play a very important part within **Integrable Systems.** However, we haven't actually defined them!



The key feature of a **conserved quantity** is that it **does not change over time**, in other words it remains constant always.

The graph to the left describes the motion of an object. What does the graph tell us?

And, what does it tell us about the energy of the whole system? Do we have enough information to make any deductions?

Considering Energy

If we are to define energy conservation mathematically we must think carefully about the energy. Whatever accounts for the energy must describe the **total energy** of that system. If it does not describe the total energy of the system we cannot think about its conversation.

In the Classical and Quantum Mechanical worlds the **total energy** of a system is described by something called the **Hamiltonian.** For our purposes, you only need to know how it is constructed from two different types of energy.

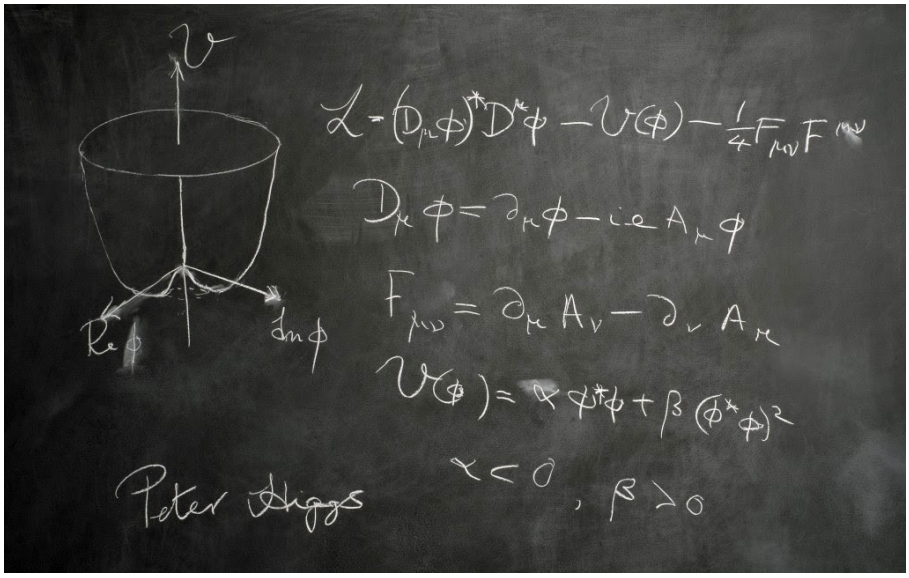
The formula that we will use to define the Hamiltonian is:

Hamiltonian = Kinetic Energy + Potential Energy,

Or simply:

$$\mathbf{H = T + V.}$$

It is a very special tool in our mathematical kit. It encodes all information regarding the way in which the system evolves over time. Magically, by design it is usually a conserved quantity automatically. If you study Maths and/or Physics at University you will find out exactly where it comes from. Both Leonhard Euler (the man with the strange hat on page 11) and Joseph-Louis Lagrange obtained incredibly important results that concern Hamiltonians, their findings are still used today in modern Physics!



The image to the right shows a Lagrangian (which is very similar to the Hamiltonian) that Peter Higgs used to describe the Higgs field, which won him and his collaborators the Nobel Prize.

Defining Conservation - Differentiation

If a **quantity** is to be **conserved**, and remain unchanged as time evolves, we must think about this mathematically and define it accordingly. We want to show that it is constant over time; namely, no rate of change. We do this by the mathematical operation of **Differentiation.**

Without realising, you have all encountered differentiation before! Cast your memories back to the baseline assessment where you asked to draw two lines governed by their own equations, of the form:

$$\mathbf{y = mx + c}$$

The gradient of a straight line is the coefficient of x, **m**. The coefficient, **m**, provides information concerning the way in which x changes over time, its rate of change. Because you know that **m** is the gradient of the line, you simply extract this from the equation of the line. In actual fact, you are differentiating!

For example, suppose you begin with the function $\mathbf{y = ax^n}$, where **a** is any number. The general rule for differentiating a function, **y**, with respect to another variable, **x**, is as follows:

$$\frac{dy}{dx} = n \times a \times x^{n-1}$$

The above simply states that you multiply the function by the original power and then subtract one from it, and this tells you how **y** changes over time with respect to **x**.

Let us now examine the equation of a straight line, **y = mx + c**. Simply substitute **n = 1** and **a=1** into the above formula, which returns

$$\frac{dy}{dx} = m.$$

The constant, **c**, simply disappears because it does not contain any dependence on **x**.

Now for some exercises, differentiate the following:

1. $y = 6x^4$
2. $y = -3x^{12}$
3. $y = x^{1/2}$
4. $y = -15x^{-2/3}$
5. $y = 4x^{-1/4}$
6. $y = \frac{1}{2}x^{-2/5}$

Defining Conservation Continued

Now that we have introduced differentiation, we can move on to consider the definition of energy conservation. It uses a slightly more complicated process of differentiation, but we will work through many examples to ensure we all understand the process.

Previously, we introduced a function: $\mathbf{y = x^n}$, which does not contain any dependence on time. If we are to show that a quantity, described by some function, is conserved by differentiation we must include the function's dependence on time.

For example consider the following more complicated function:

$$\mathbf{y = x^2(t).}$$

The above equation means that **y** is a function of **x**, and **x** is dependent on time, **t**. So as time increases the function will be effected. The tricky part of the story is that we don't know how the function depends on time, **t**! The good news is that we don't care! We just need to make sure that we appropriately acknowledge that the function is dependent on time. However, we know how the function is dependent on **x**, that is given by $\mathbf{x^2}$, and this is enough. But, when we differentiate now things are slightly different.

$$\dot{\mathbf{y}} = \frac{\mathbf{dy}}{\mathbf{dt}} = \mathbf{2x(t)} \times \frac{\mathbf{dx}}{\mathbf{dt}} = \mathbf{2x(t)} \times \dot{\mathbf{x}}$$

As you can see, different notation is used. A time derivative is denoted by a dot above the original function, in this case it is $\dot{\mathbf{y}}$. It might seem a little strange, because we have still used the previous rule for differentiation. However, the important difference is that we are looking at how the function changes over time with respect to time. The difficult part is recognising that we are differentiating with respect to a variable that does not appear explicitly in the function. We only know the **x - dependence** and this is not enough, because we know only that **x** is **time dependent**, so to make up for that we must differentiate as we would normally and multiply by the time derivative of **x**. This guarnatees that we have appropriately accounted for **x**'s time dependence.

Now pause for a moment and write any questions that you might have below, or write some notes of your own that might help you understand this.

Defining Conservation Continued

At last, we are ready to define a conserved quantity.

Definition of a Conserved Quantity:

> *A quantity, **y**, is said to be conserved if its time derivative is exactly zero, by which we mean:*

$$\dot{y} = \frac{dy}{dt} = 0.$$

A very famous theorem, **Noether's Theorem**, states that to each symmetry of your system there exists a corresponding conserved quantity. This theorem is vitally important in mathematical physics and is used a lot by researchers.

Higher Order Derivatives

As you can imagine, we can differentiate over and over again until the function no longer contains any dependence on the variable that we are differentiating with respect to. In this short course, we will only need to consider second order derivatives.

Let us again consider $\mathbf{y = ax^n}$. We have differentiated this once before, with respect to x, to obtain the first derivative:

$$\frac{dy}{dx} = n \times a \times x^{n-1}$$

Its second derivative is obtained by applying the rule again, which returns:

$$\frac{d^2y}{dx^2} = n \times (n-1) \times a \times x^{n-1-1}$$

The above represents the second derivative of function **y** with respect to **x**. If we want to differentiate a function with respect to time, we apply the same rules again as well as another (the product rule), and add an extra dot above the differentiated function to signify the second derivative. We will calculate $\ddot{y}$ together below:

$$\ddot{y} = \frac{d^2y}{dt^2} =$$

If we revisit the function $\mathbf{y = x^2(t)}$ what is its second derivative?

Homework 3 - Calculating derivatives

To make sure that we all understand how to find first and second derivatives of functions, please calculate both the first and second derivatives, with respect to **x,** of the following functions:

Hint: Just differentiate each term separately and keep track of your plus and minus signs.

1. $y = 6x^3 + 3x^2 - 2x + 7$

2. $g = x^5 - 12x^3 + 30x^2 - 12x$

3. $z = 3x + 4 + 2x^{-1} - 4x^{-2} + 5x^{-3}$

4. $h = 2x^{1/2} + 4x^{9/4}$

Differentiate the following functions with respect to **t**. Calculate their first derivative:

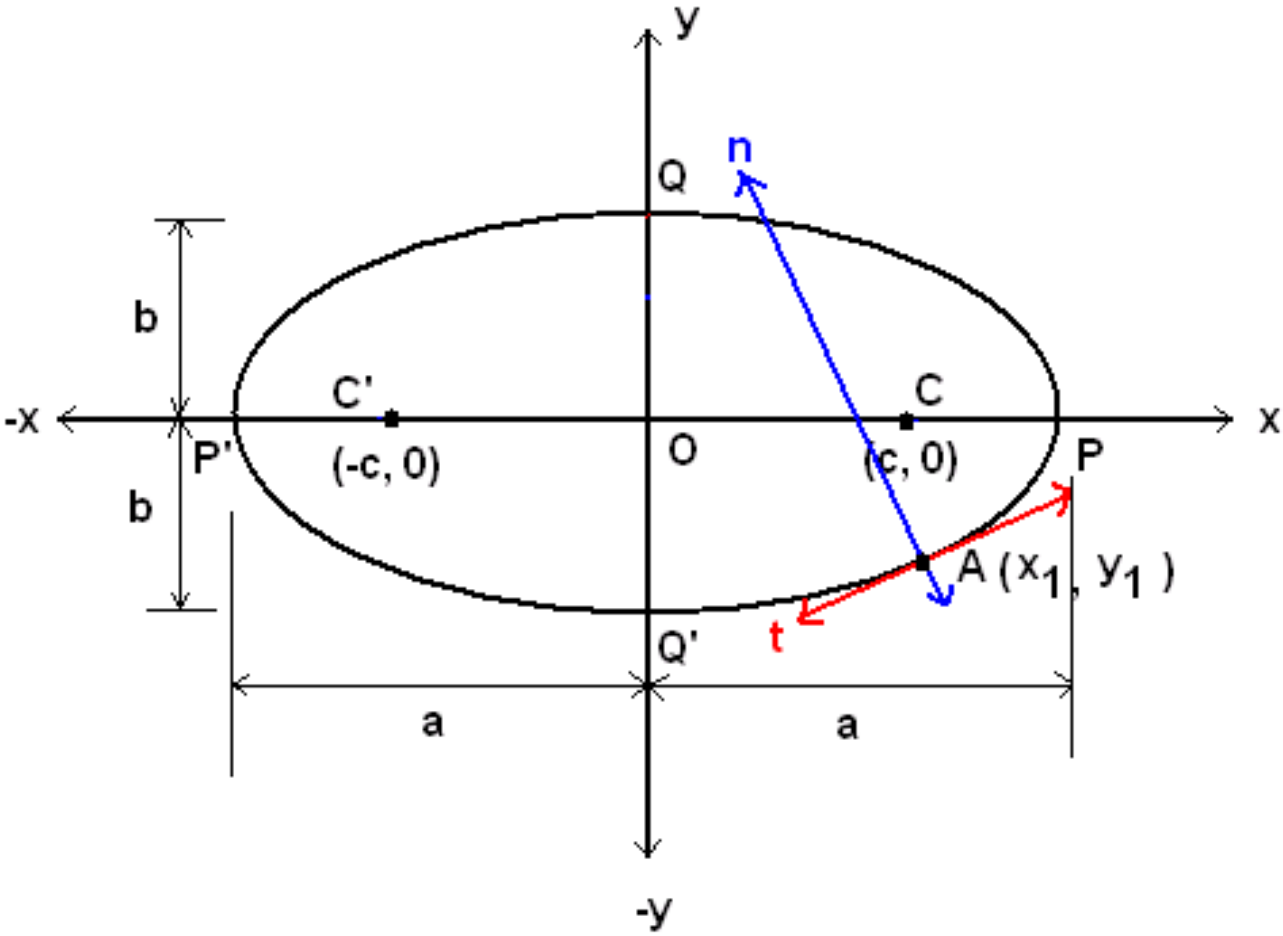
5. $b = 4x^3(t) + x^{-1/2}(t)$

6. $o = 12x(t) - 4x^2(t)$

7. $f = \frac{4}{5}x^5(t) + x^2(t) - 3x^{2/3}(t)$

8. $k = x^{12}(t) - 50x(t) + x^{-10}(t)$

Tutorial 4 – Consolidation

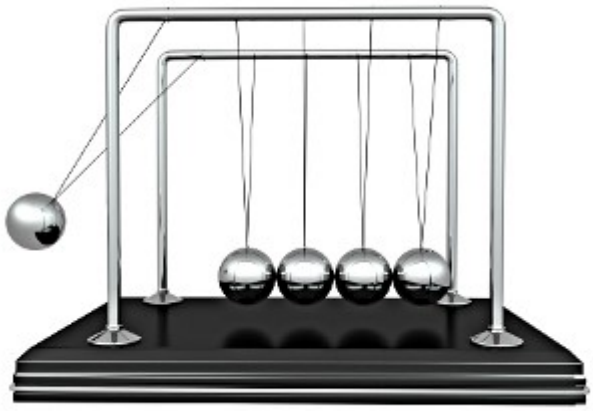


What is the Purpose of Tutorial 4?

- To go over the previous homework and address any issues.

- To understand the operation of differentiation fully. Demonstrate this by being able to calculate first and second derivatives with respect to different variables.

- To introduce a system and how you would show it has a conserved quantity.



Consolidation

We are now going to work through the homework and answer any questions you might have regarding differentiation.

We will use pages 29 and 30 to make notes as well as detail our working.

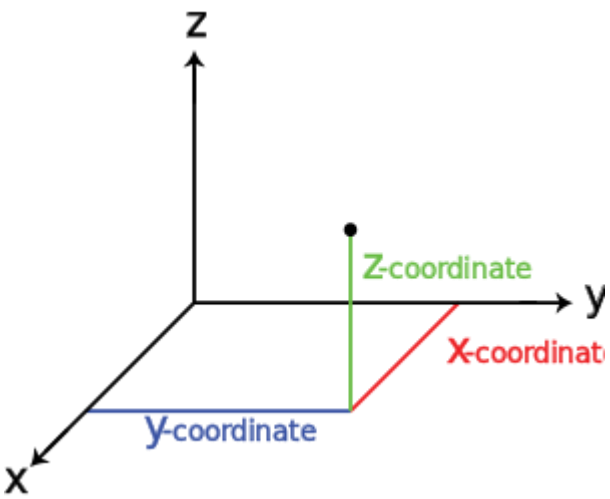
More Consolidation

Returning to Conservation

How does all of this combine to enable us to show that a quantity is conserved?

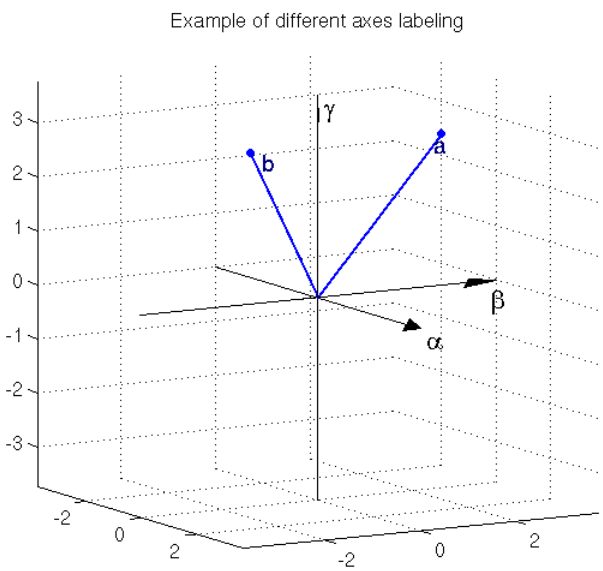
Throughout the Academic world scholars often need to know many neat little tricks that might help to simplify their problem, which will eventually allow them to solve it. This is very true in the Mathematical Physics world! We must use everything available to us that might make our lives easier.

Before we do this, we must begin to think about a real physical system and its properties. In the real world, it is likely that a system belongs to a three-dimensional space.



Sometimes x, y and z are replaced by x_1, x_2 and x_3. It doesn't matter, after all it is just notation. Everything will be fine as long as we are consistent throughout and keep track of all variables and their meaning.

We must also remember that the axes may need to include negative numbers too. The axes in the picture to the right are labelled by the Greek letters: α, β, γ, (alpha, beta and gamma).



Returning to Conservation (Continued)

We will now examine a very neat trick that helps you to discover conserved quantities.

Let us consider a system with the following Hamiltonian defined over a two-dimensional space with variables $\mathbf{x_1}$, $\mathbf{x_2}$:

$$H(x_1, x_2) = \frac{1}{2}m\dot{x}_1^2(t) + mgx_1(t)$$

Do you notice anything unusual about the Hamiltonian? Look at the variables of the system very carefully, does it look like something is missing?

This is a handy trick that can help us to identify conserved quantities!

Returning to Conservation (Continued)

Complicated Systems of Several Variables

The Hamiltonians and equations governing systems are often quite complicated. They contain several functions of more than one variable, one such example is:

$$y = x_1(t)\dot{x}_2(t)$$

And we will need to differentiate such a function with respect to **t**. But how do we do this?

As we work through this together make notes below. Clearly write each step in the process.

Homework 4 - More differentiation

Using the notes that we made in the previous tutorial, calculate the first order time derivative of the following functions:

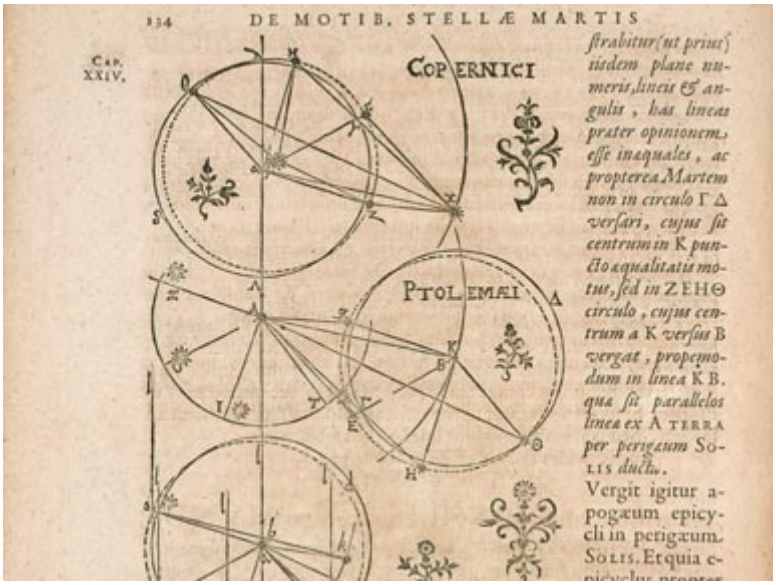
1. $S = x_1(t) + x_2(t)\dot{x}_1(t)$

2. $G = 2x_1^2(t)$

3. $W = 4x_1(t)x_2^2(t)$

4. $C = 3x_1^2(t)\dot{x}_2(t)$

Tutorial 5 – Working with Kepler

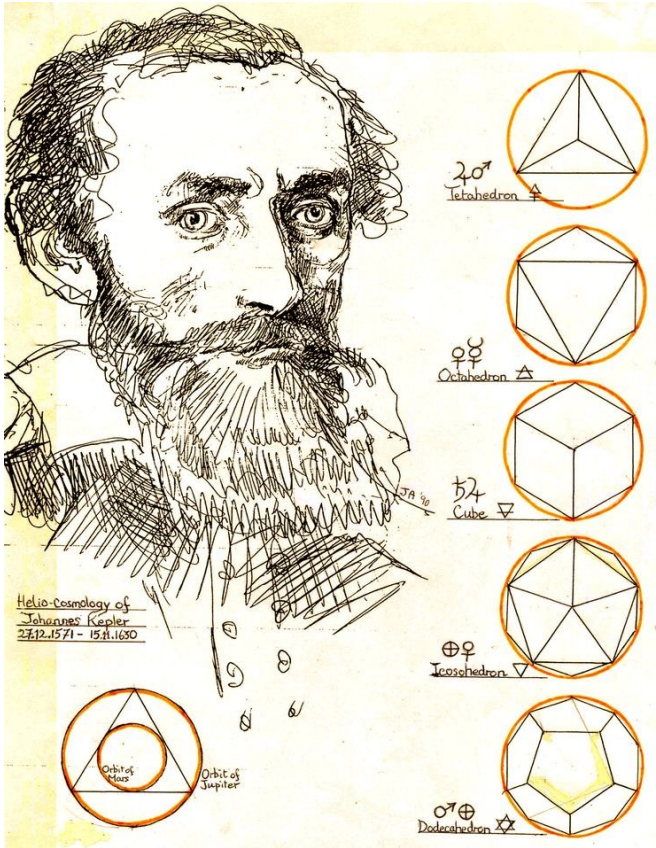


What is the Purpose of Tutorial 5?

- To go over the last homework to make sure that we know how to take first order time derivatives
- To put all of this into practice, to show that a particular quantity resulting from Kepler's laws is actually conserved.
- To introduce the final assignment.

Returning to the last homework. We will work through the answers, make notes below.

A Conserved Quantity within Kepler's Work



Once again we are working with Kepler!

This time we will present a mathematical formalism that combines everything we have covered so far. We aim to show that a quantity is conserved.

First of all, write down the definition of a conserved quantity:

The Kepler Problem

We are still dealing with an example of an Integrable system. However, the Kepler problem is a special case of a two-body problem (the two bodies could be planets for example), in which the two bodies interact via a central force (a force who's size only depends on the distance between the two bodies) that varies in strength according to the inverse square of the distance separating them.

In the space below, we will draw a diagram to accompany this description.

A Conserved Quantity within Kepler's Work (Continued)

Defining the Kepler Problem

This problem is three-dimensional, as the planets and Sun (as well as all others) are described, **spatially**, by three axes: $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$, as well as time, **t**.

The defining relations are:

$$\frac{d^2x_i}{dt^2} = -\frac{\partial}{\partial x_i} C \times (x_1^2 + x_2^2 + x_3^2)^{-1/2} \quad \text{, for all } i=1, 2, 3$$

$$\text{Hamiltonian, } H = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + C \times (x_1^2 + x_2^2 + x_3^2)^{-1/2}$$

Where C is just a constant and does not depend on time, **t**, or the spatial coordinates, $\mathbf{x}_i$.

We now want to use the skills that we have developed over the course of the tutorials to show that the following quantity is conserved:

$$J_{12} = mx_1\dot{x}_2 - mx_2\dot{x}_1$$

To do this we will use the following result:

$$\frac{\partial}{\partial x_i} C \times (x_1^2 + x_2^2 + x_3^2)^{-1/2} = -Cx_i \times (x_1^2 + x_2^2 + x_3^2)^{-3/2}$$

It is also instructive to rename $(x_1^2 + x_2^2 + x_3^2)^{-1/2} =: r$, to make our calculation look tidier.

We will now detail each step in the process and work through this together:

A Conserved Quantity within Kepler's Work (Continued)

Detailed working (continued)

Note the meaning of the following symbol for each i= 1, 2, 3. It is called the partial derivative with respect to $\mathbf{x}_i$. We will define it below:

$$\frac{\partial}{\partial x_i}$$

And so, we have at last proved mathematically that a physical quantity belonging to a system is conserved! Give yourselves a pat on the back! We will now talk through the final assignment and what you should be aiming for. Feel free to make notes in the space below.

Homework 5 - The Final Assignment

1. Prove that:

$$\frac{\partial}{\partial x_i} C \times (x_1^2 + x_2^2 + x_3^2)^{-1/2} = - Cx_i \times (x_1^2 + x_2^2 + x_3^2)^{-3/2}$$

Explain each step that you make in your working.

Hint: If you become stuck, look up the chain rule for partial derivatives and read your earlier work concerning differentiation.

2. <u>Short Research Project.</u> (Maximum 750 words)

Research Solitons, which are a fascinating feature of some Integrable systems. Make sure that you include a brief recount of:

- Their history (look into John Scott Russell's Great Wave of Translation).
- Their definition (several exist as there are different types of soliton). Clearly state at least one definition and discuss that particular type of soliton.
- Why are they such special objects?

Please include references. For instructions about referencing, see Appendix 1 on page 41. Your Solitons Research Report should be no longer than 750 words. You might find it helpful to draw diagrams to help illustrate your ideas. The picture on the front cover is actually a soliton!

3. <u>Mathematical Report.</u> (Maximum 750 words)

Write a short Mathematical Report about Integrability. Make sure you include:

- What your understanding of Integrability is.
- A brief history of the subject.
- Common features of Integrable Systems.
- Why it is helpful in the world of scientific research.

You might find it helpful to include your findings about Solitons within several of the above sections.

Tutorial 6 – Feedback tutorial

What is the Purpose of Tutorial 6?

- To receive feedback on final assignments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.

To reflect on the programme including what was enjoyed and what was challenging.

Final assignment feedback

What I did well…	What I could have improved on…

My target for future work is…

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?

What did you find challenging about the programme?	How did you overcome these challenges?

Appendix 1 – Referencing correctly

When you get to university, you will need to include references in the assignments that you write, so we would like you to start getting into the habit of referencing in your Brilliant Club assignment. This is really important, because it will help you to avoid plagiarism. Plagiarism is when you take someone else's work or ideas and pass them off as your own. Whether plagiarism is deliberate or accidental, the consequences can be severe. In order to avoid losing marks in your final assignment, or even failing, you must be careful to reference your sources correctly.

What is a reference?

A reference is just a note in your assignment which says if you have referred to or been influenced by another source such as book, website or article. For example, if you use the internet to research a particular subject, and you want to include a specific piece of information from this website, you will need to reference it.

Why should I reference?

Referencing is important in your work for the following reasons:

It gives credit to the authors of any sources you have referred to or been influenced by.
It supports the arguments you make in your assignments.
It demonstrates the variety of sources you have used.
It helps to prevent you losing marks, or failing, due to plagiarism.

When should I use a reference?

You should use a reference when you:

Quote directly from another source.
Summarise or rephrase another piece of work.
Include a specific statistic or fact from a source.

How do I reference?

There are a number of different ways of referencing, and these often vary depending on what subject you are studying. The most important to thing is to be consistent. This means that you need to stick to the same system throughout your whole assignment. Here is a basic system of referencing that you can use, which consists of the following two parts:

1. A marker in your assignment: After you have used a reference in your assignment (you have read something and included it in your work as a quote, or re-written it your own words) you should mark this is in your text with a number, e.g. [1]. The next time you use a reference you should use the next number, e.g. [2].

2. Bibliography: This is just a list of the references you have used in your assignment. In the bibliography, you list your references by the numbers you have used, and include as much information as you have about the reference. The list below gives what should be included for different sources.

Websites – Author (if possible), title of the web page, website address, [date you accessed it, in square brackets].

E.g. Dan Snow, 'How did so many soldiers survive the trenches?', http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [11 July 2014].

Books – Author, date published, title of book (in italics), pages where the information came from.

E.g. S. Dubner and S. Levitt, (2006) *Freakonomics*, 7-9.

Articles – Author, 'title of the article' (with quotation marks), *where the article comes from* (newspaper, journal etc.), date of the article.

E.g. Maev Kennedy, 'The lights to go out across the UK to mark First World War's centenary', Guardian, 10 July 2014.

Appendix 2 – Using the VLE

VLE username	
VLE password	

Please remember the following key details:

You are able log into the VLE either through the link on our website (**www.thebrilliantclub.org**) or going directly to the VLE site at (https://portal.thebrilliantclub.org/sign-in).

- Please update your profile with your full name and email address- this will allow you to retrieve forgotten passwords or usernames

If you forget your log-in details you can request them to be emailed to you by clicking the link on the VLE home page. (If you are still having problems you can email: **schools@thebrilliantclub.org**)

What is the VLE?

The VLE is a virtual learning environment for all pupils on Uni Pathways it is used for:

messaging your tutor

submitting homework

submitting your final assignment

accessing resources for your tutorials

finding out more information about university and careers

How should I use the VLE?

The VLE is a professional academic environment in which pupils are able to message their PhD Tutor. Here are a few things to consider:

Ensure you keep a professional tone in the messages you send to your tutors.

Ensure you always reply to your tutors in a timely manner.

Thank your tutor for the effort they are putting in to give you your feedback etc.

Submit all homework to your tutor on time.

IMPORTANT: Final assignment

- When you submit your final assignment, please remember that you need to do so through the **'My Activities'** tab and not as an attachment to a message.

Notes

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