Population growth: extinction or explosion?

Key Stage 4 Programme – 2016-2017

Teacher Name: Dr. L. Bowden

Pupil Name:

Timetable and Assignment Submission

Fill this in with your teacher.

Timetable – Tutorials

Tutorial	Date	Time	Location
1			
2			
3			
4			
5			
6			

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1	Baseline test	
Tutorial 2	A problem set on the logistic model	
Tutorial 3	A worksheet on differentiation	
Tutorial 4	Draft final assignment – Part one	
Tutorial 5	Final assignment	

Assignment Submission – Lateness and Plagiarism

Lateness	
Submission after 3.30pm on	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

KS4 Programme 2016-17 – Pupil Feedback Report

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-Level
2:1	60-69	Performing to a good standard at A-Level
2:2	50-59	Performing to an excellent standard at current key stage
3rd	40-49	Performing to a good standard at current key stage
Working towards a pass	0-39	Performing below a good standard at current key stage
Did not submit	DNS	No assignment received by The Brilliant Club

Name of RIS Teacher	Dr L. Bowden		
Title of Assignment	The human population – extinction or explosion?		
Name of Pupil			
Name of School	King Alfred's Academy		
ORIGINAL MARK / 100		FINAL MARK / 100	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the teacher should give an explanation in this section:

Learning Feedback Comment 1 –	
What you did in relation to this Key Learning Priority	How you could improve in the future
Learning Feedback Comment 2 –	
What you did in relation to this Key Learning Priority	How you could improve in the future
Learning Feedback Comment 3 –	
What you did in relation to this Key Learning Priority	How you could improve in the future
Resilience Comment	
How you showed learning resilience during the course	How you could build learning resilience in the future

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Course Rationale – Discrete modelling of population growth

Mathematical biology is a fast growing interdisciplinary area in which mathematical concepts are applied to many problems in the life sciences such as evolution, growth, developmental biology and medicine. Recent advances in experimental biology have enabled for detailed information about many biological processes providing the knowledge to develop accurate mathematical models. Mathematical models can be used to probe questions and make predictions about many biological processes in a cost and time effective manner.

This course introduces the concepts of developing and analysing discrete population models. Research into the dynamics of ecological populations allows us to understand the conservation of endangered species, make predictions about the effects of global climate change and understand the population fluctuations observed in the natural world. Work on infectious diseases clearly has important applications to public health, allowing us to predict the spread of an epidemic (such as Foot-and-Mouth or SARS virus) and determine the effect of control measures.

This course will explore some of the fascinating mathematics essential for building and analysing mathematical models of populations of living creatures. Material in this course is exciting and challenging, representative of the advanced mathematics lectured at university-level.
Students will use existing knowledge and skills in algebra and graph sketching to represent population dynamics. New techniques will be learnt to analyse discrete population models and determine the overall stability of a population. Students will be able to answer questions such as, 'Under what conditions will a population become extinct?' At the end of this course students will have gained skills to enable them to develop, analyse and critique simple mathematical models and interpret the results with respect to the real-life problem. These skills are vital across all areas of applied mathematics and are particularly crucial in mathematical biology and medicine.

Genesis 1:28

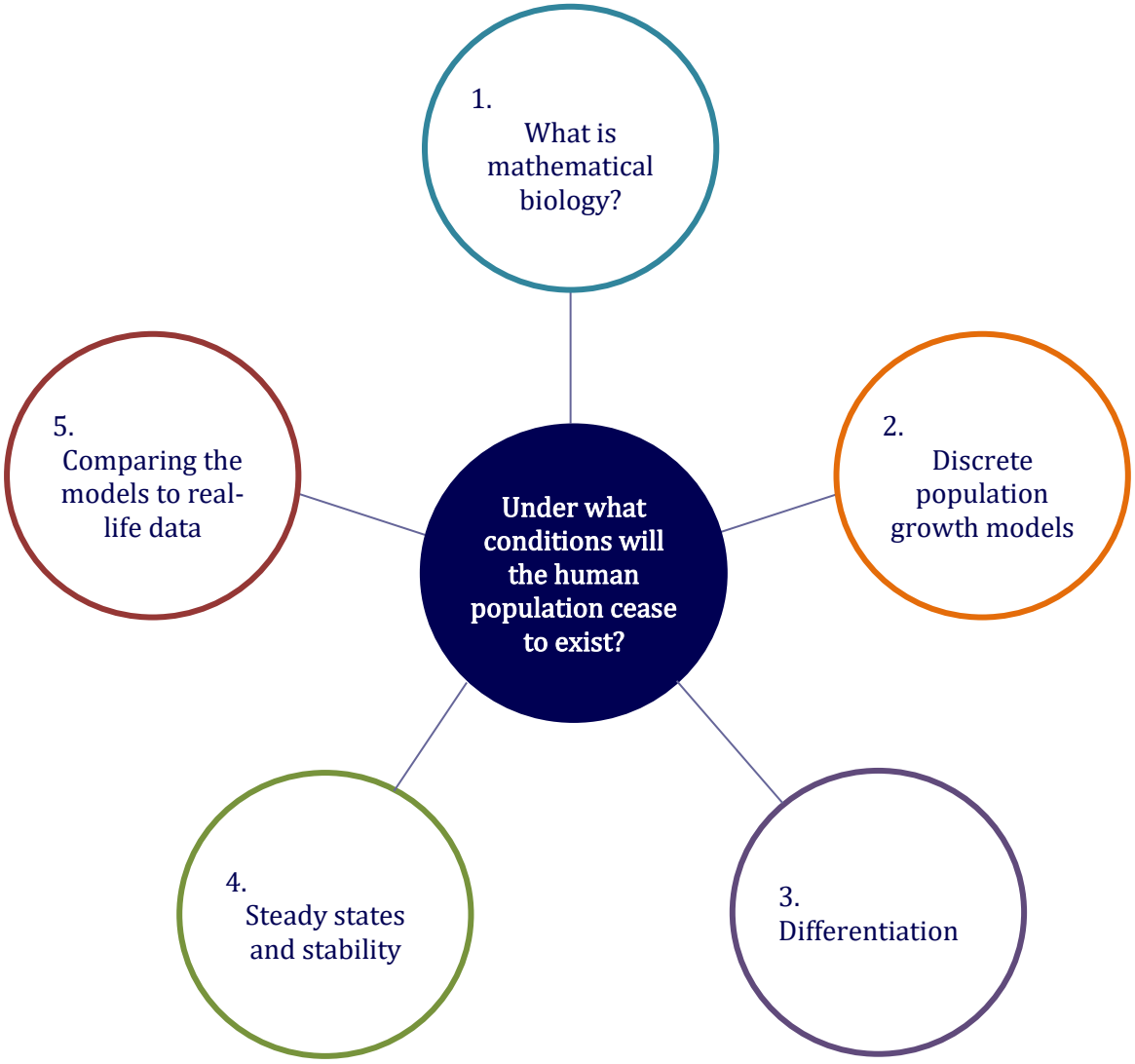
God said to them:

"Be fruitful, multiply, fill the earth, and subdue it. Have dominion over the fish of the sea, over the birds of the sky, and over every living thing that moves on the earth."

Sir David Attenborough

"The world's population is increasing out of control. If we don't do something, the natural world will."

Course Map


1. What is mathematical biology?
2. Discrete population growth models
3. Differentiation
4. Steady states and stability
5. Comparing the models to real-life data
Under what conditions will the human population cease to exist?

Mark Scheme

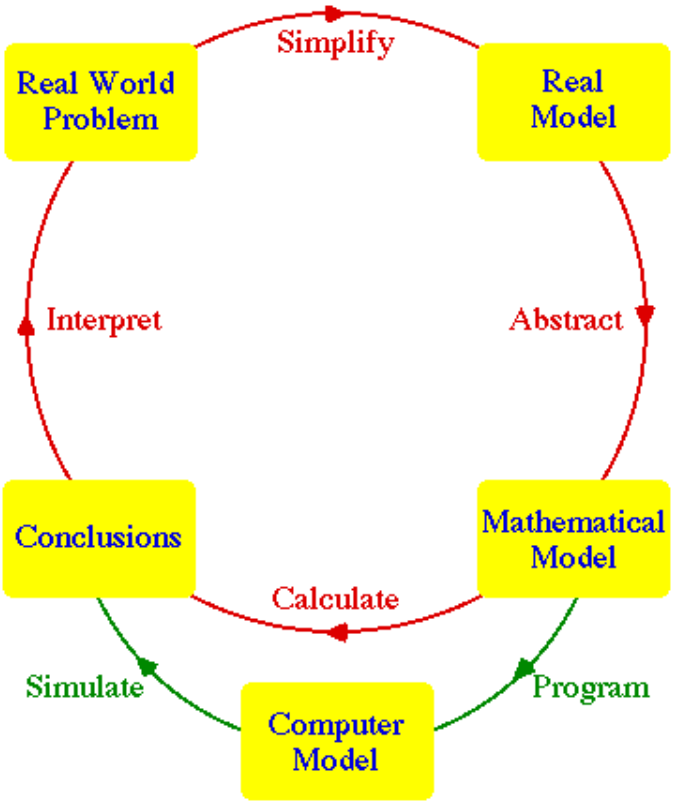
Key Skill	1st	2.2
Calculations	☐ Simple calculations are consistently correct ☐ Difficult calculations are well attempted and sometimes correct ☐ Calculations are approached logically and well presented ☐ Fluency in simulating the models in Microsoft Excel	☐ Simple calculations are well attempted and often correct ☐ Difficult calculations are attempted ☐ Calculations are well presented ☐ Simulation of the mathematical models are attempted in Microsoft Excel
Biological Interpretation	☐ Links between the biological application and the mathematical formulation are explained clearly ☐ Graphical representations can be explained clearly ☐ Can interpret the mathematical results in the context of the biological application	☐ Links between the biological application and the mathematical formulation are recognised ☐ Some understanding of the graphical representation ☐ Links between the mathematical solution and the biology are vague or missing
Terminology and Notation	☐ Correct and consistent use of mathematical and biological terminology ☐ Consistent use of sensible mathematical notation ☐ Correct and consistent sketching and labelling of graphs	☐ Some use of correct mathematical and biological terminology ☐ Some lack of understanding of the mathematical notation ☐ Some graphs without labels and annotation
Critical thinking	☐ Can recognise and explain the assumptions and limitations of a model ☐ Can discuss strengths of the model ☐ Can discuss model extensions and improvements or alternative methods ☐ Can suggest future experimental or mathematical work to test model outcomes	☐ Some attempt to discuss the strengths and weaknesses of a model

Glossary of Keywords

Throughout the course you will have opportunities to fill in the glossary with keywords and their definitions. You can refer to these words during tutorials and when completing your assignments.

Word	Definition	In a sentence…

Tutorial 1 – What is a mathematical model and mathematical biology?



Today's Key Questions:

- What is mathematical modelling? What is a model?
- How can it be used to advance our knowledge in biology?
- Which mathematical techniques that we already know could be applied to biology?

What is the Purpose of Tutorial 1?

- To understand the concept of mathematical modelling in the life sciences.
- Learn about some of the successes in the area, current research and future possibilities.
- Gain an understanding of how to build a model.

Today's Homework is:

- Baseline Assessment

The homework is due:

Tutorial content:

Mathematics and Biology

Activity 1

On the Venn diagram provided, place the topic cards in the regions you think they belong. Discuss your decisions with your partner and, where possible, justify your answers.

If there are any words you don't understand, place them outside the Venn diagram and we will discuss them at the end of the task.

What is a mathematical model?

Modelling is the use of mathematical tools and techniques as an aid to understanding situations arising outside mathematics. Constructing and studying a mathematical model involve translating the situation into mathematical terms by making assumptions, introducing notation and terminology, then studying the resulting mathematical system using mathematical concepts and methods. Next, the conclusions of the mathematical study are used to make predictions and answer questions about the original situation. Finally, these predictions are compared with observations of the original system. If the predictions are consistent with observations and provide new information about the system, then the modelling process has been useful.

Modelling can help us:

1. Further scientific understanding
2. Test the effects of changes in a system
3. Aid decision making

When constructing a model we need to think carefully about what the quantities are that we are interested in and how we can incorporate the information we already have about the system. Let's have a look at a simple example.

Example

You graduate in the year 2020 and get a job paying £1,000 per week. You have £500 per week expenses and the bank pays interest at 10% per annum. Your bank also charges £10 per month for running the account. How much will be in the account at the end of five years if you start off with nothing?

We will construct a model to solve this problem.

Problem solving versus mathematical modelling:

In the example we solved a problem by writing a mathematical formula and working out the solution. However, the process of mathematical modelling is translating the real life problem into a mathematical one through a series of assumptions and idealisations.

Activity 2

Sketch a velocity-time graph of what you see (whiteboards).

In the space below, briefly summarise our discussion of problem solving and mathematical modelling.

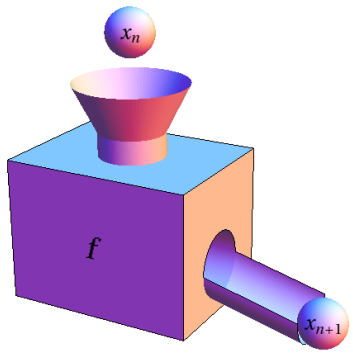
Activity 3

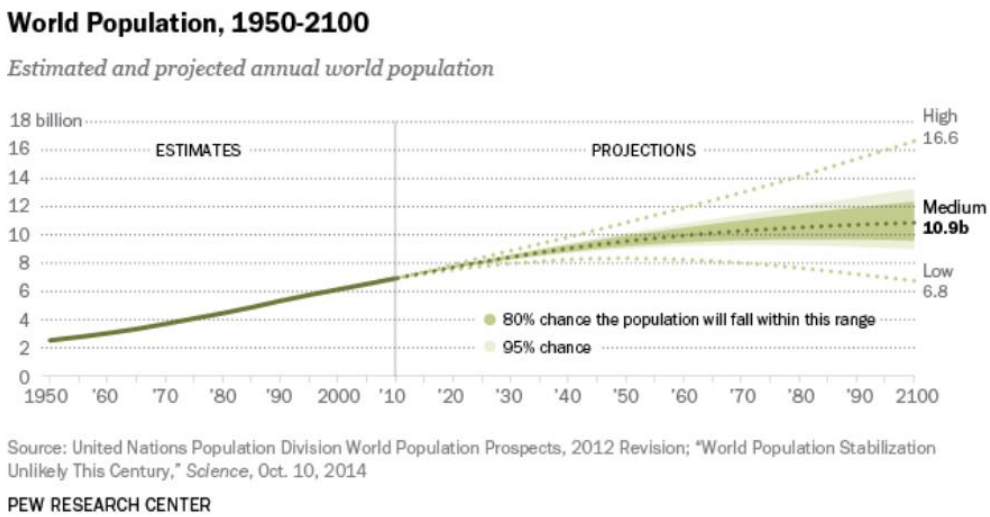
This summer you will graduate from your Brilliant Club course. You open a graduation card from a friend to find they have given you a gift of a pair of rabbits. The condition is, that any offspring your rabbits have, must be donated, in pairs, to other people in your community. They, in turn, must do the same.

By the time you graduate from King Alfred's Academy, how many people will benefit from this gift?

Notes

Tutorial 2 – Discrete population growth models





Quick Recap Quiz:

1. Mathematical modelling is…
   a) Formulating a mathematical problem from a real-life problem
   b) Strutting down the cat-walk branding the most fashionable mathematical equation

2. Mathematical modelling can assist in solving problems in
   a) Physics
   b) Chemistry
   c) Biology
   d) Engineering

Today's Key Question(s):

- How can we represent growth mathematically?
- What happens to the population for long times?
- Is the model a good description of the real biological system?

What is the Purpose of Tutorial 2?

- To understand what a discrete population growth model is.
- To be able to interpret the dynamics of a biological system from a discrete population model.
- To be able to criticise a model and recognise its limitations.

Today's Homework is:

- A worksheet on a discrete population model
- Watch http://www.mathtutor.ac.uk/differentiation/differentiationfromfirstprinciples/video in preparation for next week's tutorial.

The Homework is due:

Tutorial content:

To describe the behaviour of a population we introduce a function, f, which maps the current population size, to the population size after a given time period. Time is divided into a series of discrete steps of equal length. The length of the time step depends on the species being studied and should take into account the reproductive age of the species. For example, if we were modelling the growth of a bacterial culture, the appropriate time step may be 1 day.

To represent population dynamics mathematically we introduce the following notation:

- t represents time
- N_t is the size of the population at a given time, t
- N_{t+1} is the size of the population at time t+1
- N_0 is the initial size of the population, i.e. at t=0
- f is a function that maps N_t to N_{t+1}

Assuming that, during each time interval, the population size changes due to birth and death events only, then we can write a word equation for the population size after one time step:
(population size at time t+1) = (population size at time t) + (number of births) – (number of deaths).

Mathematically, we write the following difference equation:

$$N_{t+1} = N_t + B - D$$

Where B and D represent the number of births and deaths in one time step, respectively.

Activity 1

What happens to the population for:

1. $B > D$

2. $B = D$

3. $B < D$?

Birth and death rates

It is intuitive to relate the number of births and deaths to the population size.

Activity 2

$$b = \frac{B}{N}$$

Justify why we will relate the number of births and deaths to the population size.

The birth rate per capita is defined as the number of births per individual per unit time:

Similarly, the death rate per capita is defined as the number of deaths per individual per unit time.

Activity 3

1. Write down the form of the death rate.

2. Substitute the birth and death rates into the difference equation and simplify to show that our equation can be written as

$$N_{t+1} = (1+r)N_t, \quad \text{where} \quad r = b - d$$

The rate r is the geometric rate of increase. We can define λ = 1 + r, as the rate of change of population size, so that the difference equation can be written as

$$N_{t+1} = \lambda N_t$$

This is a linear difference equation where the rate of change in the population size is proportional to the size of the population; therefore the rate λ is density independent. We call this a simple exponential model (you will see why in Activity 6) and is often referred to as the Malthusian growth model after Thomas Robert Malthus. Malthus was famous for his insights into population growth and the author of An Essay on the Principle of Population (1798).

Activity 4

Read the information below and answer the questions.

In 2015, the population of the red fox was estimated to be around 250,000. The red fox is a seasonal breeder reproducing in the winter. About 400,000 new cubs are born each year. Due to natural predation, death, disease and road accidents approximately 325,000 die.

1. What is the death rate d?

2. What is the birth rate b?

3. What is the value of λ?

4. Use our model to predict the population size in 2020.

Activity 5

Consider the linear difference equation $N_{t+1} = \lambda N_t$

1. What values can λ take? Explain your answer.

2. Deduce the solution for N_t in terms of the initial population N_0

3. Explain what happens to the population for different values of λ. Justify your answer with sketches of the solution for the different cases.

4. How long does it take a population to double in size? Give your answer in terms of λ.

It is important to remember that, while a mathematical model can be extremely useful, there is no correct model. Different models can be developed to represent the same biological system and each has its strengths and weaknesses. A good mathematician recognises the limitations of a model and under what assumptions the outcomes hold.

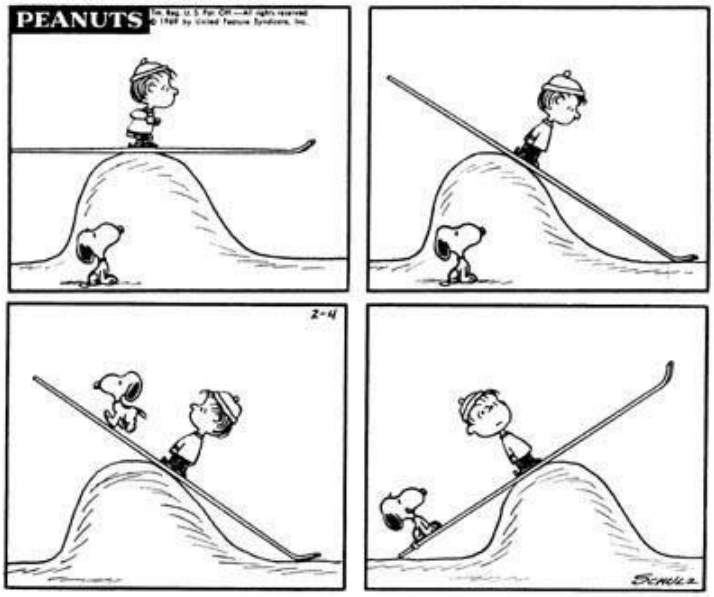
Activity 6

Discuss in your group the assumptions made when developing the exponential model and what problems it has. Think about what else could affect the size of the population?

Write down your thoughts.

Notes

Tutorial 3 – Differentiation and rate of change



Quick Recap Quiz:

1. Population ecology is the study of:
   a) Growth rate of species populations
   b) Dynamics of species populations
   c) Death rate of species populations
   d) None of the above

2. In the model $N_{t+1} = \lambda N_t$, N represents
   a) Time
   b) Food supply
   c) Population size
   d) Space

Today's Key Question(s):

- How do we find the gradient of a curve?
- What is significant about the derivative of a function in relation to population models?

What is the Purpose of Tutorial 3?

- To understand what the derivative of a function means
- To learn rules for differentiating functions
- To understand why finding the derivative is useful in the context of population models

Today's Homework is:

- A worksheet on differentiation

The Homework is due:

Tutorial content

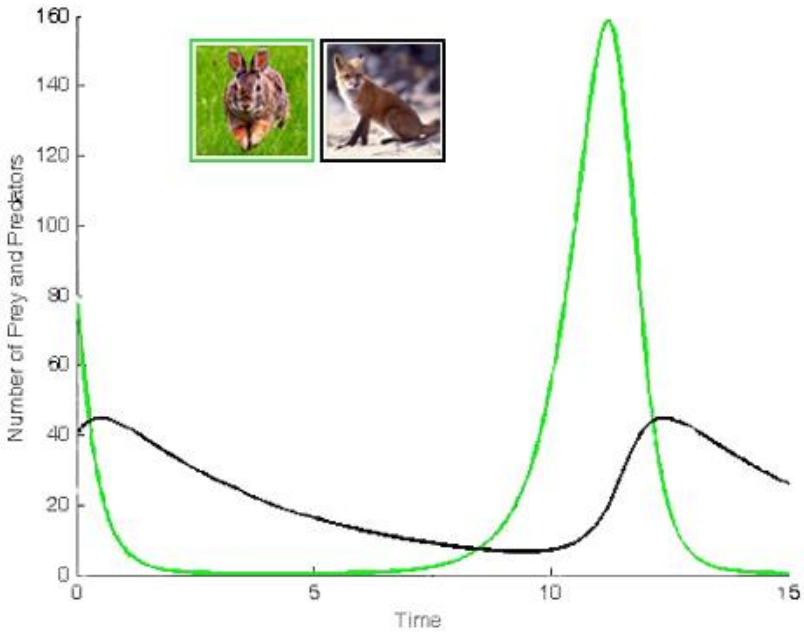
Differentiation enables us to determine how one quantity changes with respect to another, i.e. it is a tool that allows us to find the gradient of a graph. In population dynamics, this is usually how the population size changes with respect to time.

The gradient of a function $y = f(x)$ tells us what the rate of change of y is with respect to x. For linear functions (a straight line), the gradient is constant, however, for nonlinear curves the gradient changes.
To find out how fast something has travelled, knowing how it went and how long it took we can work

Activity 1

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

Explain what is happening to these populations as time evolves.



What does the gradient of these curves represent?

out

On a distance-time graph this is equivalent to working out the gradient.

What happens when the graph is not a straight line?

The gradient of a curve $y = f(x)$ at a given point, is equal to the gradient of the tangent to the curve at that point.

Activity 2

1. Sketch the curve $y = x^2$

2. Work out the gradient at several points by drawing tangents to the curve. Can you spot a pattern?

Let's try and find the gradient at a general point.

Activity 3 – fill me in

Call the general point A with coordinates:

Let B be a point close to A with coordinates:

Then the gradient of the line AB is:

Activity 4

Repeat Activity 3 for $y = x^3$ and $y = x^4$

Fill in the table below with your results.

y	gradient
x^2	
x^3	
x^4	
x^n	

Can you predict what the gradient at a given point on the curve $y = x^n$ will be?

The gradient of a curve is given by the derivative of the function and is $f'(x)$ denoted

Rules of differentiation

Rule 1:

To differentiate $f(x) = x^n$,

- bring the existing power down and use it to multiply
- reduce the old power by one and this becomes the new power

The derivative is $f'(x) = nx^{n-1}$

Rule 2:

To differentiate $f(x) = x$,

- note that $x = x^1$
- differentiating gives $1x^0 = 1$

The derivative is $f'(x) = 1$

Rule 3:

To differentiate $f(x) = a$,

- a constant always differentiates to zero

The derivative is $f'(x) = 0$

Rule 4:

To differentiate $f(x) = ax^n$,

- bring the existing power down and use it to multiple
- reduce the old power by one and this becomes the new power

The derivative is $f'(x) = anx^{n-1}$

Rule 5:

To differentiate $f(x) \pm g(x)$,

- differentiate $f(x)$ and $g(x)$
- add or subtract the results

The derivative is $f'(x) \pm g'(x)$

Activity 5

Differentiate the following functions:

1. $f(x) = x^2$
2. $f(x) = x^8$
3. $f(x) = 3x$
4. $f(x) = \frac{1}{2}x$
5. $f(x) = 6$
6. $f(x) = 101$
7. $f(x) = 2x^4$
8. $f(x) = 6x^7$
9. $f(x) = \frac{1}{2}x^2 + 3x$
10. $f(x) = 2x^6 + 9x^4 + 7x + 1$
11. $f(x) = \frac{1}{x}$
12. $f(x) = \frac{2}{x}$
13. $f(x) = \frac{7}{x^2}$
14. $f(x) = x^{\frac{3}{2}}$
15. $f(x) = 4x^{\frac{1}{4}}$
16. $f(x) = \sqrt{x}$

The derivative and population models

In the context of population models, the derivative tells us how fast a population is increasing or decreasing. This can be particularly important in the area of population control when external factors can influence the rate of change of a population size. For example, determining the percentage of the population that requires vaccination or achieving a maximum harvest without causing extinction.

The Product Rule

The derivative of the product of two functions is given by

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

Example

Differentiate $x^2(1+2x^4)$.

Let $f(x) = x^2$ and $g(x) = 1 + 2x^4$ then, $f'(x) = 2x$ and $g'(x) = 8x^3$.
Using the product rule,

$$\begin{aligned}(f(x)g(x))' &= f'(x)g(x) + f(x)g'(x)\\ &= 2x(1+2x^4) + x^2(8x^3)\\ &= 2x(1+2x^4) + 8x^5\\ &= 2x(1+2x^4+4x^4)\\ &= 2x(1+6x^4).\end{aligned}$$

This could have been been calculated by expanding out the brackets first.
Check this way gives the same result:

The Quotient Rule

The derivative of the quotient of two functions is given by

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

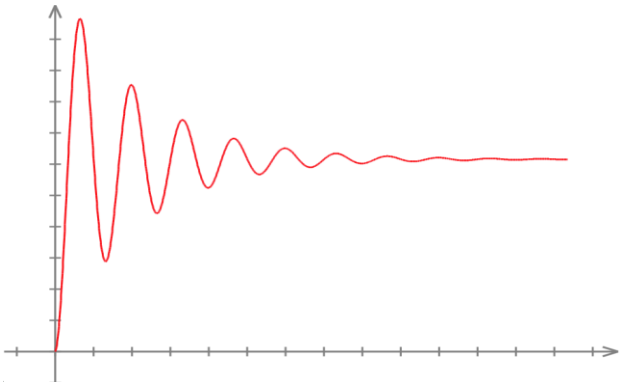
Example

Differentiate

$$\frac{2x}{x^2+1}$$

Notes

Tutorial 4 – Steady states and stability



Quick Recap Quiz:

1. The rate of change of a quantity is
   a) How much the quantity changes with respect to space
   b) How much the quantity changes with respect to time
   c) The speed
   d) How much it changes with to volume

2. The rate of change of a graph is given by
   a) The y-intercept
   b) The x-intercept
   c) A coordinate
   d) The gradient

3. To find the rate of change of a quantity we do what to a function?
   a) Solve it
   b) Differentiate it
   c) Exterminate it
   d) Multiply it

What is the Purpose of Tutorial 4?

- To calculate steady states of the system.
- To understand the meaning of stability and instability.
- To deduce the stability of a population using differentiation.

Today's Key Question(s):

- Will the population go extinct?
- Is there a non-negative steady state?
- Is the steady state stable?

Today's Homework is:

- Draft Final Assignment (Part one)

The Homework is due:

Tutorial content

If the number of individuals remains constant over time, the population is said to have reached a steady state. For example, in a closed population, if the number of births balances the number of deaths then the size of the population will remain constant.

We can define this mathematically as follows:
A population N* is a steady state, if and only if, $N^* = f(N^*)$.

The 'if and only if' in this statement means that the implication works both ways. If N* is a steady state then $N^* = f(N^*)$. Similarly, if $N^* = f(N^*)$ then N* is a steady state. This is an important point because not all statements are reversible like this. For example, *if a number is divisible by 9, it is divisible by 3*, is a true statement. The reverse, however, is false, *if a number is divisible by 3, it is divisible by 9.*

Activity 1

For the following discrete population models, find all the steady states and comment on the parameter restrictions for these steady states to exist.

1. $N_{t+1} = rN_t$

2. $N_{t+1} = rN_t + \beta$

3. $N_{t+1} = \frac{rN_t}{N_t + A}$

4. $N_{t+1} = \frac{rN_t^2}{N_t^2 + A}$

5. $N_{t+1} = N_t e^{r\left(1 - \frac{N_t}{K}\right)}$

Stability of a steady state

If a steady state is stable, it means that any small perturbation from this point will result in the population returning to the steady state.
Mathematically we can work this out by finding the derivative:
A steady state, N*, is stable if and only if $|f'(N^*)| < 1.$
If a steady state is not stable, we say it is unstable.

Activity 2

For Questions 1 to 4 in Activity 1, state whether the steady states are stable or unstable. Where stability depends on the parameter values, state the restrictions.

Activity 3

The following model has been used to model fish populations (Beverton and Holt, 1957)

$$N_{t+1} = \frac{\alpha N_t}{1 + \beta N_t}, \quad \text{where } \alpha \text{ and } \beta \text{ are positive numbers and } N_t \geq 0.$$

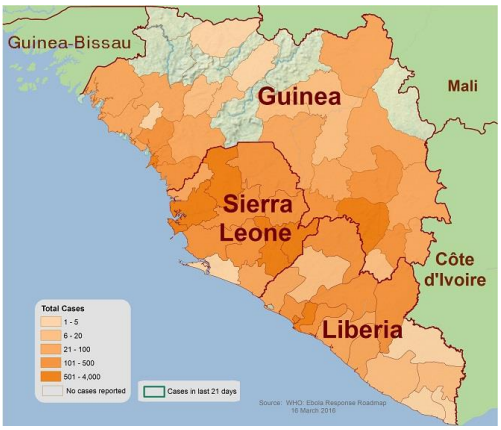
a) Determine the steady states. For each steady state give the range of α and β for which it exists.

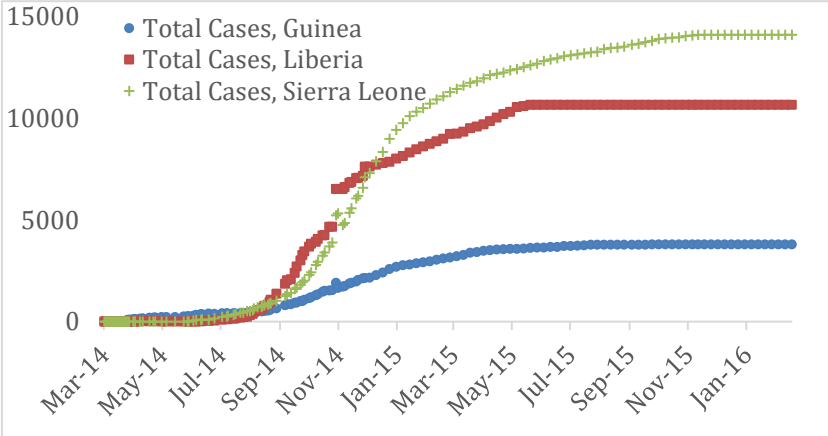
b) Indicate whether the steady states are stable or unstable and state whether the behaviour near the steady state is monotonic or oscillatory.

R.J.H Beverton and S.J. Holt, (2012) 'On the dynamics of exploited fish populations' (Volume 11), Springer Science & Business Media

Notes

Tutorial 5 – Using real-life data





Quick Recap Quiz:

1. A population has reached steady state if
   a) It is able to balance on one leg
   b) $N^* = f(N^*)$
   c) The number of individuals remains constant over time
   d) Time stops

2. A steady state is stable if
   a) The population returns to the steady state after a small perturbation
   $|f'(N^*)| > 1.$ b)
   $|f'(N^*)| < 1.$ c)
   $|f'(N^*)| = 1.$ d)

3. The behaviour of the solution near the steady state is oscillatory if
   a) The derivative of the function is negative
   b) The derivative of the function is positive
   c) The derivative of the function is zero
   d) The individuals sway back and forth in a regular rhythm

Today's Key Question(s):

- Do the models accurately represent the real-life problem?

What is the Purpose of Tutorial 5?

- Learn how to use an inbuilt solver in Excel to fit a model to data
- Determine whether the model predictions provide valuable insight
- Consider model refinements based on comparisons with data

Today's Homework is:

- Final Assignment

The Deadline is:

Tutorial Content

In this tutorial we will look at a data set from the 2014 West Africa Ebola outbreak.

- We will adopt two of our discrete population models and discuss what the variables and parameters represent in this real life case.
- We will compare our models to the data using a method of error minimisation.
- We will discuss the conclusions of the models and how these translate to the real life case.

Background

At the end of 2013, the most widespread epidemic of the Ebola virus broke out in West Africa. The outbreak began in Guinea and spread to the neighbouring countries of Liberia and Sierra Leone. It is not entirely clear how the initial infection occurred. For the purpose of this tutorial we will be studying the rate at which the Ebola virus spread and how it came to be under control.

Modelling

At the time of the outbreak, the World Health Organisation would have called upon mathematical modelling to inform their decisions. It would have been of great importance to determine the rate at which the Ebola virus was spreading across West Africa. With accurate predictions of the rate of spread of the virus, the WHO would be able to ascertain effective prevention measures and whether it would be necessary to quarantine infected individuals.

Activity 1

Let's consider two discrete population models:

1. the exponential model $$N_{t+1} = \lambda N_t$$
2. the logistic model $$N_{t+1} = rN_t\left(1 - \frac{N_t}{K}\right)$$

a) What do the variables and parameters represent in the models?

b) Can we say anything about an of the parameter values now we know what they represent biologically?

Data fitting

To fit the models to the data we are going to use Excel. This allows us to input the data, the models and use an inbuilt solver to perform the fitting for us. To find the best fit of the model to the data we are

going to vary the parameters of the model. The inbuilt solve will calculate the error between the model solution and the data curve for different parameter values. This is called a least squares optimisation and aims to minimise the mean squared error (MSE) given by the following expression:

$$\mathrm{MSE} = \sum_{i=1}^{k} (x_i - N_i)^2$$

At each point in time, i, the difference between the data value, x_i, and the value given by the model, N_i, is calculated and squared. This gives a measure of the error of our model at each time point. These errors are then summed over all time points giving the MSE.

Activity 2

Use the Excel spread sheet provided to investigate how well the exponential model fits the data. Include a graph of your best fit, the parameter values, a prediction of the model and a conclusion of whether the model is sensible.

Activity 3

Use the Excel spread sheet provided to investigate how well the logistic model fits the data. Include a graph of your best fit, the parameter values, a prediction of the model and a conclusion of whether the model is sensible.

Notes

Final Assignment: The human population – extinction or explosion?

Warm up exercises (see Appendix 2)
⇒ These are the things you will need to have mastered to complete part one of your final assignment.

Part one
⇒ A guided problem set displaying the mathematical skills and techniques learnt in the course.
⇒ Part one will be set as the draft assignment after tutorial 4.

Part two
⇒ Data analysis of the given data set using excel.
⇒ Mathematical analysis and critique of the suggested model drawing on the biological and mathematical assumptions, the strengths and the limitations.
⇒ Suggestions of better versions of the model with explanation and justification of the changes made.

Part one

In Tutorial 2 we looked at the exponential model. Here we consider exponential growth of a population with migration. The model is as follows:

$$N_{t+1} = rN_t + \beta$$

1. Explain what each of the terms, parameters and variables represent in the model. Include the following in your explanation:
   a) What do N, t, r and β represent?
   b) Are N, t, r and β dependent/independent variables or parameters?
   c) For what values of N, t, r and β is the model realistic?
   d) What are the dynamics of the population for different values of r and β ?

2. a) Find the steady states of the model.
   b) Discuss the restrictions on the parameters for a positive steady state to exist.
   c) Explain why, if it exists, the steady state should be positive.
   d) Discuss what is happening biologically when the population reaches steady state.
   e) Under what conditions is the steady state stable?

*3. Show, by induction (see Appendix 2) or otherwise, that the solution to the model is given by:

$$N_t = \left(N_0 - \frac{\beta}{1-r}\right) r^n + \frac{\beta}{1-r}$$

Part two

We now consider modelling human population dynamics using a discrete-time model.

- The first question will draw on the skills acquired in Tutorial 5. You will be asked to analyse the data set by fitting suggested mathematical models to it and drawing conclusions about the data and about the suitability of the models.
- Question two is similar to part one. You will be asked to analyse the models proposed using mathematical techniques learnt in Tutorials 2, 3 and 4.

- For the final question you will be asked to discuss the assumptions of the models and whether you think the predictions made are reliable. If you have alternative ideas of what a model for human population growth should look like, please suggest changes and justify your equations.

1. The data given in Tables 1 and 2 are the human population sizes from 1 AD to 2016.

Table 1

Year (AD)	Population (billions)
1	0.188
100	0.195
200	0.202
300	0.205
400	0.209
500	0.210
600	0.213
700	0.226
800	0.240
900	0.269
1000	0.295
1100	0.353
1200	0.393
1300	0.392
1400	0.390
1500	0.461
1600	0.554
1700	0.603
1800	0.990

Table 2

Year (AD)	Population (billions)
1900	1.65
1910	1.75
1920	1.86
1930	2.07
1940	2.30
1950	2.52
1960	3.02
1970	3.68
1980	4.44
1990	5.31
2000	6.13
2010	6.93
2011	7.01
2012	7.10
2013	7.18
2014	7.27
2015	7.35
2016	7.43

a) Which data entries are not as you would expect? Using the internet, research what could have caused this ambiguity in the data. Reference your sources.

b) Use the data to argue that, for some periods of time, an exponential model would be a good representation of population growth. You may want to consider the following techniques:
   a. Plotting the data on a graph
   b. Using different data points to calculate the growth rate
   c. Comparing the data curves to an exponential curve.

c) Now argue why the exponential model may not be a good representation of population growth. What does the exponential model suggest will happen to the human population in the future?

d) The data suggest that, in recent history, the growth of the population has been slowing down. By plotting the growth rate against time since 1900, show that this is true and identify approximately when this began.

2. We now assume that human population growth can be modeled by the following:

$$N_{t+1} = rN_t\left(1 - \frac{N_t}{K}\right)$$

   a) What do N, t, r and K represent?
   b) Are N, t, r and K dependent/independent variables or parameters?
   c) For what values of N, t, r and K is the model realistic?
   d) Discuss the dynamics of this model and justify why is may be a good model for the human population data.
   e) Find all the steady states and discuss what they mean biologically including any parameter restrictions.
   f) Determine under what conditions the steady states are stable/unstable.

3. Discuss the limitations of the model suggested in Question 2. You may want to consider the following:
   a) Assumptions of the model
   b) Interactions that the model does not account for. For example, should the human population depend on things other than the current population size?
   c) Special cases e.g. is the model realistic if the population began with $N_0 = 1$?

4. Given your considerations in Question 3. What would you include in a new model of population growth? Feel free to suggest mathematical forms for your model and/or suggestions of the important features. Discuss also, how you would validate your model.

References for Table 1 and 2

C. McEvedy and R. Jones, (1978) 'Atlas of World Population History', Facts on File, New York, pp.342-351.
J.N. Biraben, 'An Essay Concerning Mankind's Evolution', *Population, Selected Papers,* 1980.
J.D. Durand, 'Historical Estimates of World Population: An Evaluation', *Analytical and Technical Reports*, Number 10, 1974,

C. Haub, 'How Many People Have Ever Lived on Earth?' *Population Today*, 1995.
R. Thomlinson, (1975) 'Demographic Problems, Controversy Over Population Control', Second Edition.
United Nations, (1999) 'The World at Six Billion: World Population From Year 0 to Stabilization', pp.5.
U.S. Census Bureau (USCB), 'Total Midyear Population for the World: 1950-2050', 2012.
M. Kremer, 'Population Growth and Technological Change: One Million B.C. to 1990', *Quarterly Journal of Economics*, pp.681-716, 1993.

Notes

Tutorial 6 – Feedback

What is the Purpose of Tutorial 6?

- Reflect on skills learned on the programme with a view to encouraging resilience
- Receive feedback on final assignment

What are the key questions?

- What strengths have I demonstrated in my work and what areas of development are there left for me to address?
- What steps do I need to take to improve my academic output?

Activity 1 – Marking your own

Using the mark scheme, identify three things you did well and three things you could improve.

What went well?

1.

2.

3.

What could be improved?

1.

2.

3.

Activity 2 – Responding to feedback

Read through your feedback sheet and highlight on your assignment where you received positive feedback (one colour) and areas for improvement (different colour). You might find it helpful to annotate this with a brief comment or two.

Activity 3 – Looking Forwards: Think, Pair Share

1. Think about how you can use this feedback in your future school work.
2. Talk to your partner about how you both plan to make changes to your school work in the future as result of this feedback
3. Share your ideas as a group.

Appendix 1 – Supporting material

BBC news article – Is population growth out of control?

http://www.bbc.co.uk/news/magazine-24303537

What is mathematical biology?

http://www.ams.org/notices/201007/rtx100700851p.pdf

https://www.maths.ox.ac.uk/groups/mathematical-biology

Discrete population models (J. D. Murray)

http://matriisi.ee.tut.fi/courses/MAT-35006/pIIexam/Murray_Ch2.pdf

http://nrich.maths.org/7252

Help with differentiation

http://www.bbc.co.uk/bitesize/higher/maths/calculus/

Fitting a model to data in Excel

http://www.jkp-ads.com/articles/leastsquares.asp

Limitations to mathematical models

http://serc.carleton.edu/quantskills/methods/models/limits.html

Appendix 2 – Proof by Induction

Often we want to proof that a statement is true for all natural numbers n.
As an example we will prove, by induction, that

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Now to check this is true, we may start by testing the first few values of *n*

$$n = 1: \quad \sum_{k=1}^{1} k^2 = 1^2 = 1$$

$$\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$$

$$n = 2: \quad \sum_{k=1}^{2} k^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6} \times 2 \times 3 \times 5 = 5$$

$$n = 3: \quad \sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

But, $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6} \times 3 \times 4 \times 7 = 14$ we want to prove this for all natural numbers and therefore doing it this way is impossible.

Instead, we do the following:

1. Show that the statement is true for n = 1
2. Assume that the statement is true for n (this is called the inductive hypothesis)
3. Using this assumption, prove the statement is true for n + 1

Now, since we showed it is true for n = 1 (in step 1)
… then it is true for $n = 1 + 1 = 2$ (from step 2 and 3)
… and for n = 2 + 1 = 3
… and so on.

Let's use our example above.

1. We have already shown that the statement is true for n = 1
2. Assume that the statement is true for n, that is

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

3. Now we need to prove the statement is true for n + 1. To do this we plug n+1 into the left hand side of our statement we wish to prove. We use our inductive hypothesis that we have assumed to be true and manipulate the right hand side.

$$
\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= (n+1)^2 + \sum_{k=1}^{n} k^2 \\
&= (n+1)^2 + \frac{1}{6}n(n+1)(2n+1) \qquad \text{by the inductive hypothesis} \\
&= (n+1)\left((n+1) + \frac{1}{6}n(2n+1)\right) \\
&= \frac{1}{6}(n+1)(6(n+1) + n(2n+1)) \\
&= \frac{1}{6}(n+1)(6n + 6 + 2n^2 + n) \\
&= \frac{1}{6}(n+1)(2n^2 + 7n + 6) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3)
\end{aligned}
$$

The right hand side now looks familiar and can be rewritten to resemble the original statement for n+1

$$
\begin{aligned}
\sum_{k=1}^{n+1} &= \frac{1}{6}(n+1)(n+2)(2n+3) \\
&= \frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1)
\end{aligned}
$$

We have shown, by mathematical induction, that

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all natural numbers $n > 0$.

Appendix 3 – Referencing correctly

When you get to university, you will need to include references in the assignments that you write, so we would like you to start getting into the habit of referencing in your Brilliant Club assignment. This is really important, because it will help you to avoid plagiarism. Plagiarism is when you take someone else's work or ideas and pass them off as your own. Whether plagiarism is deliberate or accidental, the consequences can be severe. In order to avoid losing marks in your final assignment, or even failing, you must be careful to reference your sources correctly.

What is a reference?

A reference is just a note in your assignment which says if you have referred to or been influenced by another source such as book, website or article. For example, if you use the internet to research a particular subject, and you want to include a specific piece of information from this website, you will need to reference it.

Why should I reference?

Referencing is important in your work for the following reasons:

- It gives credit to the authors of any sources you have referred to or been influenced by.
- It supports the arguments you make in your assignments.
- It demonstrates the variety of sources you have used.
- It helps to prevent you losing marks, or failing, due to plagiarism.

When should I use a reference?

You should use a reference when you:

- Quote directly from another source.
- Summarise or rephrase another piece of work.
- Include a specific statistic or fact from a source.

How do I reference?

There are a number of different ways of referencing, and these often vary depending on what subject you are studying. The most important to thing is to be consistent. This means that you need to stick to the same system throughout your whole assignment. Here is a basic system of referencing that you can use, which consists of the following two parts:

A marker in your assignment: After you have used a reference in your assignment (you have read something and included it in your work as a quote, or re-written it your own words) you should mark this is in your text with a number, e.g. [1]. The next time you use a reference you should use the next number, e.g. [2].

Bibliography: This is just a list of the references you have used in your assignment. In the bibliography, you list your references by the numbers you have used, and include as much information as you have about the reference. The list below gives what should be included for different sources.

Websites – Author (if possible), title of the web page, website address, [date you accessed it, in square brackets].

E.g. Dan Snow, 'How did so many soldiers survive the trenches?', http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [11 July 2014].

Books – Author, date published, title of book (in italics), pages where the information came from.

E.g. S. Dubner and S. Levitt, (2006) *Freakonomics*, 7-9.

Articles – Author, 'title of the article' (with quotation marks), *where the article comes from* (newspaper, journal etc.), date of the article.

E.g. Maev Kennedy, 'The lights to go out across the UK to mark First World War's centenary', *Guardian*, 10 July 2014.

Appendix 4 – University Applications Guidance

For course choices and careers information:
www.brightknowledge.org.uk

For information and statistics on universities and course:
www.unistats.direct.gov.uk

For comprehensive information on universities, including rankings:
www.thecompleteuniversityguide.co.uk

PLEASE NOTE – The Researchers in Schools programme is designed to support the university applications process at your school. The work you complete for the programme should add to your schoolwork rather than detract from it. Please be aware of deadlines and try to manage your workload appropriately. If you are worried about the programme interfering with your schoolwork then please speak to your teacher.

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