Multiplication in a world without numbers

Key Stage 4 Programme

Pupil Name

Handbook Designed by Dr Oliver King

Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1			
2			
3			
4			
5			
6 (Feedback)			

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1	Space, Dimensions and Algebras	
Tutorial 2	Permutations and Diagrams	
Tutorial 3	Generators	
Tutorial 4	The Brauer algebra	
Tutorial 5	Diving deeper into the Brauer algebra	

Assignment Submission – Lateness and Plagiarism

Lateness	
Submission after midnight on	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

The Brilliant Club KS4 Programme – Pupil Feedback Report

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-level
2:1	60-69	Performing to a good standard at AS-level
2:2	50-59	Performing to an excellent standard at GCSE
3rd	40-49	Performing to a good standard at GCSE
Working towards a pass	0-39	Performing below a good standard at GCSE
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Name of Teacher	Dr Oliver King		
Title of Assignment	Multiplication in a world without numbers		
Name of Pupil			
Name of School	Tibshelf Community School		
ORIGINAL MARK / 100		FINAL MARK / 100	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the teacher should give an explanation in this section:

Learning Feedback Comment 1 - ***Understand what a mathematician might want to know about an algebraic object***

What you did in relation to this Key Learning Priority	How you could improve in the future

Learning Feedback Comment 2 – ***Learn to spot patterns and possibly justify them***	
What you did in relation to this Key Learning Priority	How you could improve in the future

Learning Feedback Comment 3 – ***Become comfortable with a new, abstract calculation method***	
What you did in relation to this Key Learning Priority	How you could improve in the future

Resilience Comment	
How you showed learning resilience during the course	How you could build learning resilience in the future

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Course Rationale

Research mathematics can broadly be divided into two areas: pure and applied. Whereas applied maths will deal with the real world and look at ways of using maths to explain what we see, pure maths is much more abstract, often solving problems simply out of curiosity. In this course we will take a look at some abstract mathematical objects called algebras. You may understand our world as a 3-dimensional space (possibly 4 if you include time), but algebras often have many more dimensions than this, and also have some kind of symmetry baked into them. While we will be examining these from the perspective of a pure mathematician, so looking for interesting properties rather than applications, these algebras do turn up in many places. For instance, modern physics uses a looks at the world using a toolkit called "string theory", where instead of having particles make up the universe we have tiny strings (and I mean tiny, these are only 10^{-35} metres long!). This idea turns our universe into an 11-dimensional space and we understand this by studying the so-called Virasoro algebra, which is defined purely abstractly. Indeed, many areas of physics these days boil down to studying algebras rather than performing experiments in the lab (some physicists nowadays never set foot inside a lab!).

Since these algebras are often impossible to visualise entirely (just try to imagine what a 4D cube looks like, let alone an 11D one!), mathematicians find ways of breaking them up into easier pieces and studying those, before gluing them back together again. We will use a few of these techniques to study a class of algebras called "diagram algebras". As the name suggests, the definition of these is in terms of pictures, and much of the work we will do is very hands on. (Also, as the title suggest, there will be very few numbers involved!) The study of diagram algebras is a very active area of research, with dozens of papers being published every month.

This course aims to give you a look at what a professional mathematician does when doing research, an insight into the kind of questions they might ask, and some of the strategies we use to answer these. You'll have a head start into the world of mathematics and theoretical physics, and will learn how to construct logical, mathematical arguments that are useful in many more areas of study.

Mark Scheme Table

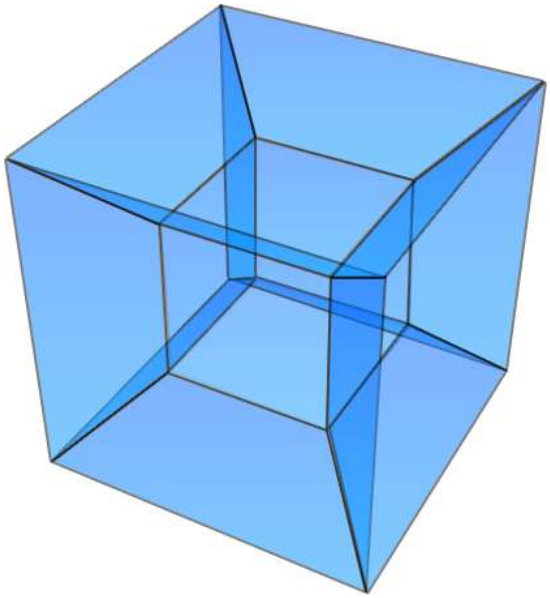
	Skills	1st (70-100)	2:1 (60-69)	2:2 (50-59)
Subject knowledge	Knowledge and Understanding	○ All content included is relevant to the general topic and to the specific question/title ○ Good understanding of all the relevant topics. ○ Scientific terms are defined and used accurately throughout ○ Clear justification on how the content included is related to the specific issues that are the focus of the assignment	○ Most of the content included is relevant to the general topic and to the specific question/title ○ Good understanding of most the relevant topics ○ Scientific terms are used accurately but not always clearly defined. ○ Adequate justification on how the content included is related to the specific issues that are the focus of the assignment	○ Some of the content included is relevant to the general topic and to the specific question/title ○ Good understanding on some of the relevant topics but occasional confusion on others. ○ Scientific terms are used mostly accurately with occasional confusion and often not defined. ○ Some justification on how the content included is related to the specific issues that are the focus of the assignment
Critical thinking	Research and Evidence	○ Use evidence/calculations to support claims/assertions/ ideas, consistently clearly and convincingly	○ Use evidence/calculations to support claims/assertions/ideas, mostly clearly and convincingly	○ Use evidence/calculations to support claims/assertions/ideas, at times clearly and convincingly
	Developing an Argument	○ Argument/proof exceptionally well-developed and well-justified ○ Uses concepts from the tutorials in an unfamiliar context, and does so accurately and confidently.	○ Argument/proof clear and well-developed and position justified ○ Use some concepts from the tutorials in an unfamiliar context, but not always accurate	○ Argument/proof clear but not well-developed ○ Limited use of concepts from the tutorials in other contexts

	Critical Evaluation	○ Moved beyond description to an assessment of the value or significance of what is described	○ Mostly description but some assessment of the value or significance of what is described	○ Only description with minimal assessment of the value or significance of what is described
Written communication	Structure and Presentation	○ Ideas are presented in paragraphs and arranged in a logical structure that is appropriate for the assignment ○ All calculations, formulas and methods are clearly structured, clear to follow and correct.	○ Ideas are presented in paragraphs and arranged in a structure that is mostly appropriate for the assignment ○ Calculations, formulas and methods are mostly structured, clear to follow and correct.	○ Ideas are presented in paragraphs and arranged in a structure ○ Calculations, formulas and methods are not always structured, clear to follow and correct.
	Language and Style	○ No spelling, grammar or punctuation errors ○ Writing style consistently clear, appropriate for scientific documents and easy to follow ○ Accurate and consistent use of technical language and vocabulary	○ Minimal spelling, grammar or punctuation errors ○ Writing style mostly clear, appropriate for scientific documents and easy to follow ○ Some attempts of using technical language and vocab alary, but not always accurate	○ Some spelling, grammar or punctuation errors ○ Writing style moderately clear, appropriate for scientific documents and easy to follow ○ Use of simple language and vocabulary effectively but struggles to use technical language

Glossary of Keywords

Word	Definition
Cayley table	A table showing how the multiplication rules work in an algebra
Commutative	When the order of multiplication does not matter
Concatenation	The process of putting one diagram on top of another and deleting the central nodes
Generating set	A set of objects that, when multiplied together over and over, make up all others.
Horizontal arc	A line joining nodes on the same side of a Brauer diagram
Multiplicand	One of the quantities in a multiplication
Parity	Whether a number is odd or even
Permutation	The shuffling of some set of objects into an order.
Propagating arc	A line joining nodes on different sides of a Brauer diagram
Propagating number	The number of propagating arcs in a Brauer diagram
Straightening	The process of removing extra loops from a Brauer diagram

Tutorial 1 – Space, Dimensions and Algebras



What is the Purpose of Tutorial 1?

- To learn a few of the reasons we want to study Algebra
- To be happy with working in higher dimensional spaces
- To learn the idea of multiplication in an algebra

What is Algebra?

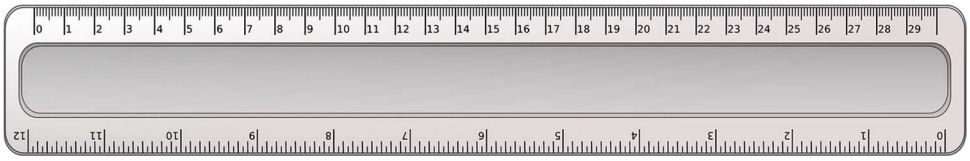
Discuss

What does a 1-dimensional space look like? A 2-dimensional space? 3, 4, 5…?

Dimensions are easier than you think

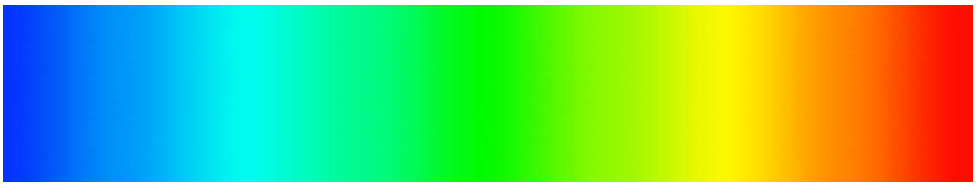
To help us understand a dimension all we need is something that changes on a scale, like how we can measure length with metres or time in seconds. For dimensions up to 3 this is easy, just think of the space we live in.

How to we describe a point in a 3-dimensional space? What about a 2-dimensional space?

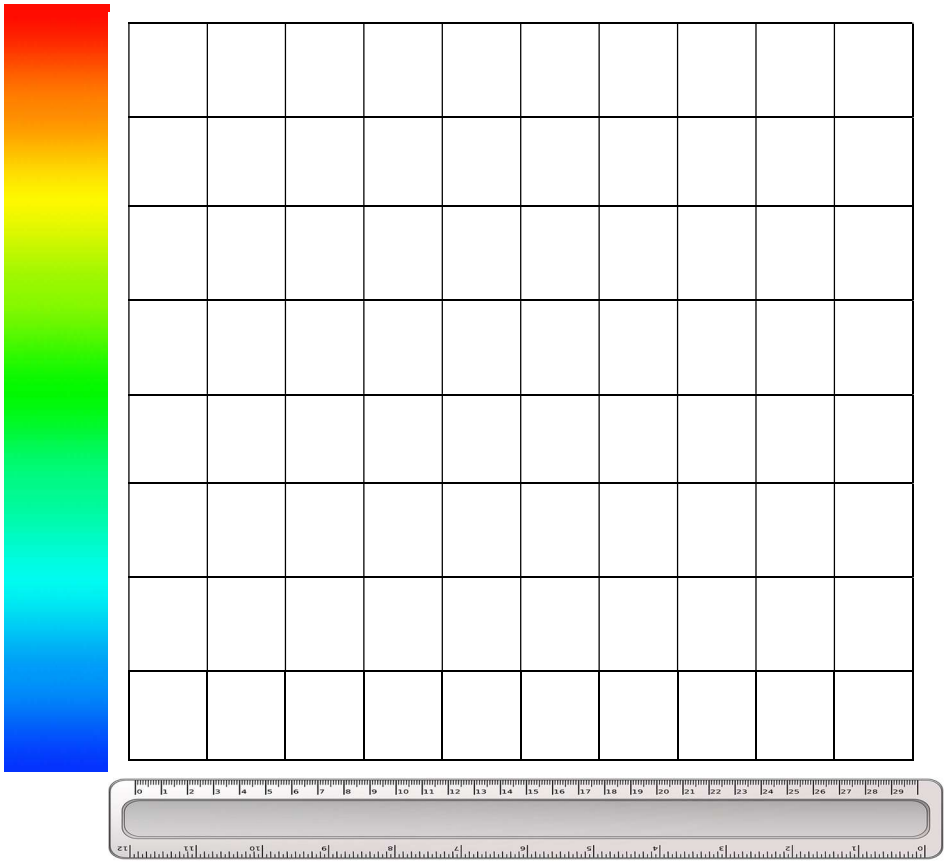


The ruler above represents a 1-dimensional space. Imagine you live on one end of the ruler and a friend lives on the other. Is it possible to both walk along the ruler and swap places?

The colour spectrum is in many ways like a ruler, but it measures colour instead of length. Imagine now that you live on the spectrum below, and wherever you stand you become that colour.



What we will do now is combine both the ruler and the colour spectrum. Imagine you are free to walk along the ruler, but also change your colour.



As we can see from the diagram above, this colour and ruler combo gives us a 2-dimensional space (like a standard xy graph). But since we don't need the extra axis with colour, we can imagine a 2D space on a 1D surface!

Now imagine again that you and your friends live on either side of the ruler, but you are both free to change colour. Can you swap places? Use the colour/ruler axes above to help you.

Activity

Try to untie some of the knots I have given you. Using the colour argument, show that any knot can be untied in 4-dimensions.

Higher dimensions

The colour argument shows that we can represent a point in a 4-dimensional by its xyz coordinate and colour, totalling 4 pieces of information. We can think of this as a coordinate with 4 entries:

$$(x, y, z, c)$$

where c is colour. So to represent an n-dimensional space, where n is any positive whole number, we just need n pieces of information to put into a coordinate.

Multiplication

If we think about algebraic expressions involving x, y, z and c, then we might see something like this:

$$2x^2 + 4yz - 3xz^3 + c - 9xyzc$$

Think about what the terms x^2, yz, xz^3 and $xyzc$ mean. There is some multiplication going on here, but since we don't know the rules we cannot work them out. Imagine trying to work out 3×3 or $2 \times 7 \times 9$ without having learned the times tables. We just need a new "times table" to tell us how to work out x^2 or y^3z. This is called a "Cayley table", named after the mathematician Arthur Cayley (1821-95):

$\downarrow \times \rightarrow$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{c}$
$\boldsymbol{x}$	x	y	z	c
$\boldsymbol{y}$	y	$-x$	c	$-z$
$\boldsymbol{z}$	z	$-c$	$-x$	y
$\boldsymbol{c}$	c	z	$-y$	$-x$

The Cayley table tells us how to multiply two things together, to do more we just do them two at a time. For example

$$\begin{aligned} xyzc &= (x \times y) \times (z \times c) \\ &= (y) \times (-x) \\ &= -yx \\ &= -y \end{aligned}$$

Now try working out a few of these multiplications:

1) $x^2 =$ _____

2) $xy =$ _____

3) $y^2c =$ _____

4) $cxz =$ _____

5) $x^2zy^3 =$ _____

Homework 1

The purpose of this homework is to become familiar with multiplying using a Cayley table.

1) Use the following Cayley Table to work out the multiplications below.

$\downarrow \times \rightarrow$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{a}$	a	b
$\boldsymbol{b}$	b	0

a) $a^2 =$ _____

b) $ab^2 =$ _____

c) $baba =$ _____

d) $a^2b^3a =$ _____

2) Use the following Cayley Table to work out the multiplications below.

$\downarrow \times \rightarrow$	$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{s}$	$\boldsymbol{t}$	$\boldsymbol{u}$
$\boldsymbol{p}$	p	q	r	s	t	u
$\boldsymbol{q}$	q	p	t	u	r	s
$\boldsymbol{r}$	r	u	p	t	s	q
$\boldsymbol{s}$	s	t	u	p	q	r
$\boldsymbol{t}$	t	s	q	r	u	p
$\boldsymbol{u}$	u	r	s	q	p	t

a) $t^3 =$ _____

b) $q^2rt =$ _____

c) $qrq =$ _____

d) $rqr =$ _____

e) $t^2up^3t =$ _____

3) A multiplication rule is "commutative" if the order in which we write the elements doesn't matter. Regular multiplication of numbers is commutative since we can do it in any order: $5 \times 7 = 7 \times 5$, $326 \times 437 = 437 \times 326$ and so on. For each of the Cayley tables below, write whether or not they are commutative and include your reasoning.

a)

$\downarrow \times \rightarrow$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{a}$	a	b
$\boldsymbol{b}$	b	a

b)

$\downarrow \times \rightarrow$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{a}$	a	$-b$
$\boldsymbol{b}$	b	0

c)

$\downarrow \times \rightarrow$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{a}$	0	0
$\boldsymbol{b}$	0	0

d)

$\downarrow \times \rightarrow$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{a}$	a	b
$\boldsymbol{b}$	a	0

4) (Optional) Can you think of a way to tell if a Cayley table is commutative simply by looking at it?

Tutorial 2 – Permutations and Diagrams

What is the Purpose of Tutorial 2?

- To understand what a permutation is
- To be able to represent permutations as diagrams
- To be able to multiply permutation diagrams

Activity

Choose two people to stand in a line. How many different orders can you stand in?

Now try the same with three people, how many this time?

What about all 4?

Think about how could we write these down without using yourselves or other objects?

Permutations

What we were doing in the previous activity was seeing how many "permutations" of 2,3 and 4 people there are. Permutation is the mathematical term for a shuffle (as in deck of cards). Permutations are found throughout mathematics, and the study of these is very important. Therefore we need a nice way to handle them.

Rather than think of shuffling people or cards around, we usually use numbers when dealing with permutations. We will call the objects by their position in the sequence (so 1 will correspond to the first

object, 2 to the second etc.), and so shuffling up the objects can be described by saying which position 1 goes to, where 2 goes to etc.

One way to view this is called "Cauchy's two-line notation", named after the mathematician and physicist Augustin-Louis Cauchy (1789-1857). In this notation, we list the objects we are shuffling in one line, and underneath we write where that object is sent. For instance

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

will send the first object to the second position, the second to the first, and the third will stay where it is. Try to describe the following permutations using words:

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

Multiplying permutations

In the previous tutorial, we learned how to multiply things together using a Cayley table. However when dealing with permutations this is not really practical – think about how big the Cayley table for permuting 4 objects would be! Thankfully permutations are nice, and we don't have to use one. What we need to do is work out the meaning of something like

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

We do this by following a number through both permutations:

So the product $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ sends 1 to 3. Fill in the rest of the two-line permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & & \end{pmatrix}$$

We can now multiply any two permutations together, without using a Cayley table. Try multiplying these together:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

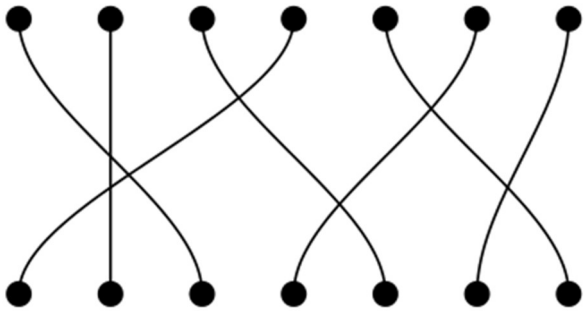
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Diagram notation

Cauchy's two-line notation is useful, but we will now learn a different way of writing permutations. We will use an example to help us. Let's take the permutation on 7 numbers below:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 1 & 7 & 4 & 6 \end{pmatrix}$$

Rather than having two lines of numbers, we have two lines of dots (called nodes). So draw 7 dots in one row, and then facing them below draw another 7. Now we join those nodes on the top to those on the bottom with curved lines. The first node on top is joined to the third node on the bottom, the second node on top is joined to the second on the bottom, and so on until the seventh node on top is joined to the sixth node on the bottom. We end up with a diagram like this:

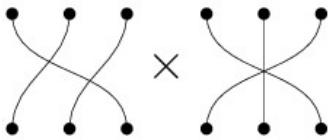


This is called a "permutation diagram". Try drawing the diagrams of the following permutations:

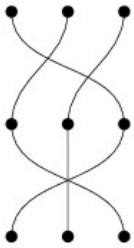
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 6 & 1 & 3 & 5 \end{pmatrix}$

Multiplying permutation diagrams

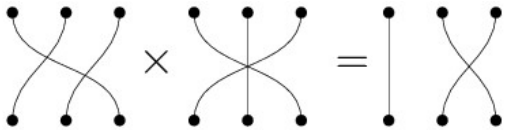
Multiplying permutation diagrams is again done without using a Cayley table. The process here is called "concatenation". We will again use an example to help explain this. To work out the product



we draw the left diagram above the right diagram so that the bottom nodes of the left overlap the top nodes of the right:



The final answer to the multiplication is given by forgetting about the nodes in the middle and considering the diagram on the top and bottom nodes. Therefore in our example:



Activity

In pairs, write down a few permutations and draw some permutation diagrams. One person try multiplying the permutations, the other try the diagrams. What are the advantages and disadvantages of each method?

Homework 2

This homework will help you become comfortable with multiplying permutations together, and converting between two-line notation and diagram notation.

1) Calculate the following products.

a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} =$ _____

b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}^2 =$ _____

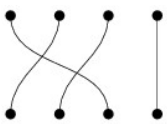
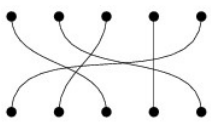
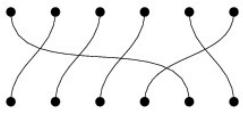
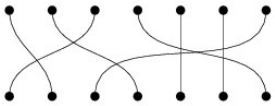
c) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} =$ _____

d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}^2 \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix} =$ _____

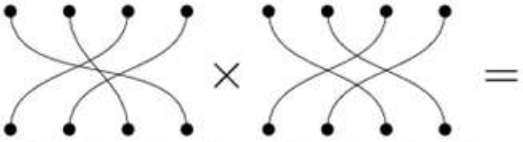
2) Convert the following two-line permutations into diagrams.

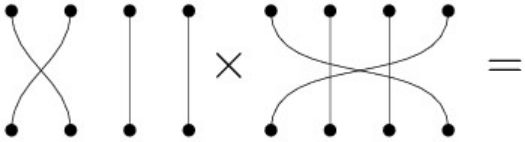
a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 5 & 3 \end{pmatrix}$	b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 5 & 2 & 3 \end{pmatrix}$
c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 1 & 4 & 3 & 6 \end{pmatrix}$	d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 6 & 2 & 3 & 8 & 7 & 4 \end{pmatrix}$

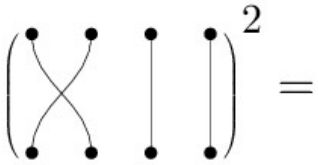
3) Convert the following diagrams into two-line permutations.

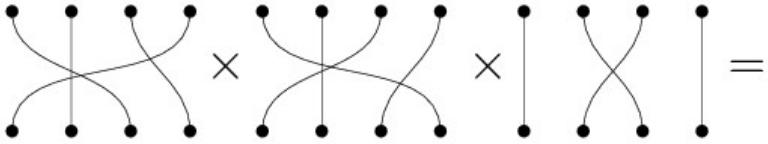
a) 	b) 
c) 	d) 

4) Calculate the following products.

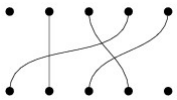
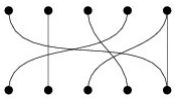
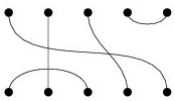
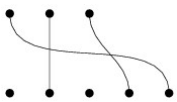
a) 

b) 

c) 

d) 

5) Which of the following are permutation diagrams? Why?

a) 	b) 
c) 	d) 

6) Is multiplication of permutations commutative? Give reasons for your answer.

Tutorial 3 – Generators

What is the Purpose of Tutorial 3?

- To understand what a generating set is
- To be able to write a permutation as a product of generators

!

Look back at the beginning of Tutorial 2. How many permutations of 2 people were there? 3 people? 4 people? How about if we want to know the number of permutations of 7 people?

So the number of permutations of n objects is

This is also written as $n!$ (say "n factorial").

Factorials grow extremely quickly. If we wanted to work with permutations on 10 numbers then we would be dealing with a 3,628,800-dimensional space! Thankfully, we don't usually need to worry about that many permutations.

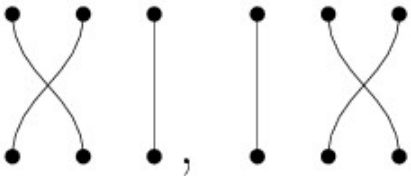
Activity

Draw a permutation diagram with two rows of three nodes (not one on the cards!). Try to recreate it using only the cards given.

Now draw a permutation with two rows of four nodes. Can you still make it using only these cards? If not, what is missing?

Generators

A "generating set" is a collection of permutations which, when multiplied together again and again, make up any permutation you want (on the same number of objects). As seen from the card activity, the pair



generate all permutations on 3 objects. Note that we only need 2 diagrams to make all 6 permutations on 3 objects.

Discuss

Can you think of any other generating sets for permutations on 3 objects? How many diagrams are in each set?

How many generators?

In general, there are many generating sets for permutations. However there are a couple that are particularly useful. The first is the "minimal generating set":

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-1 & n \\ 2 & 1 & 3 & 4 & 5 & & n-1 & n \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-1 & n \\ 2 & 3 & 4 & 5 & 6 & & n & 1 \end{pmatrix}$$

Why might this be interesting to mathematicians?

However this set is not always that easy to work with. Try using the $n = 4$ version to generate a permutation.

Another generating set is the Coxeter set, named after H. S. M. Coxeter (1907-2003). We have already seen this in the 3 object case. In general we can write it in two-line notation as:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 1 & 3 & & n-1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 3 & 2 & & n-1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 2 & 4 & & n-1 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & 3 & & n & n-1 \end{pmatrix}.$$

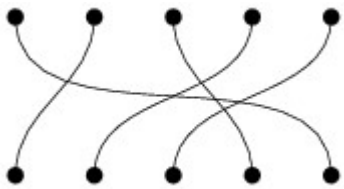
Try writing these in diagram notation below.

How many of these Coxeter generators are there?

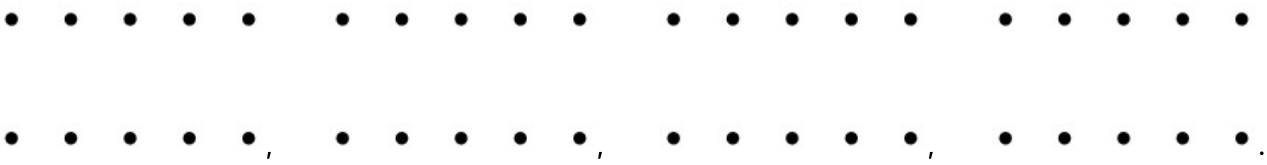
Coxeter generators are much easier to work with, look at the following example.

Example

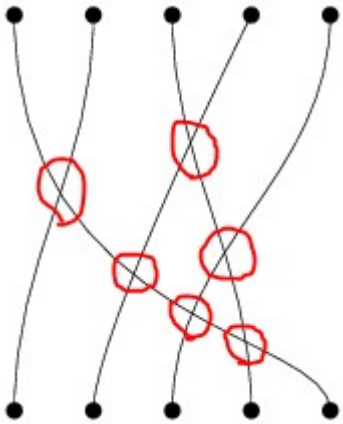
We want to write



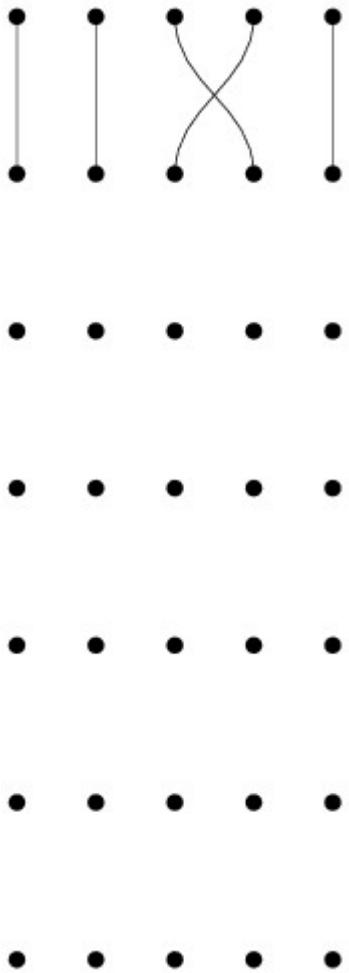
as a product of the Coxeter generators



We start by stretching out the diagram so that there isn't more than one crossing in the same row. Then highlight each crossing



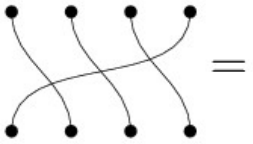
Now starting from the top crossing, we put in the appropriate Coxeter generators. Since the first crossing here is between the third and fourth node, we use the third generator above. Complete the rest of the picture on the next page.

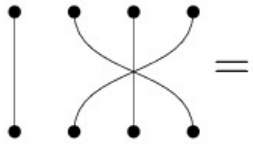


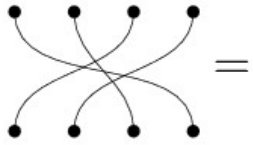
Homework 3

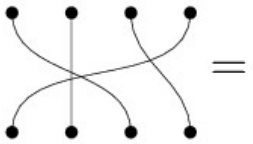
This homework sheet will strengthen your understanding of generators.

1) Write the following diagrams as a product of Coxeter generators.

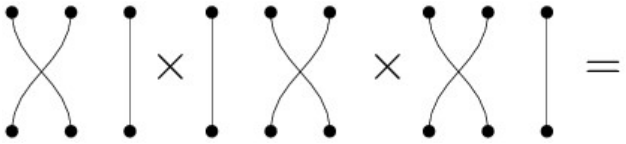
a) 

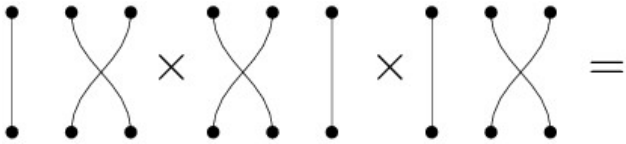
b) 

c) 

d) 

2) Calculate the following two products of Coxeter generators.

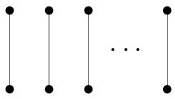
a) 

b) 

What does this tell you about the way we write a permutation as a product of generators?

3)

a) Show that when any diagram is multiplied on the left or right by

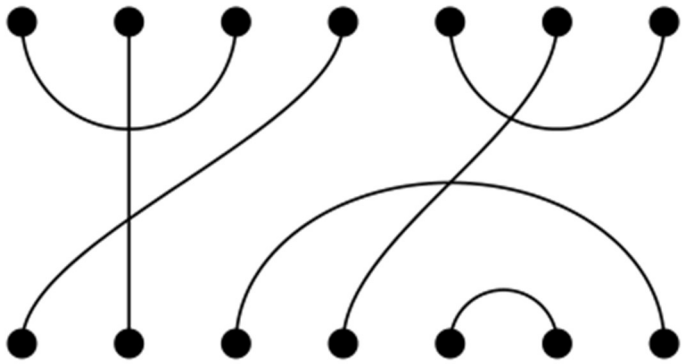


the result is the original diagram.
(Hint: you will need to argue for all diagrams, not just one or two examples. Think about what happens when you put this diagram on top or underneath another)

b) Show that any Coxeter generator multiplied by itself gives the diagram from part a).

c) Using parts a) and b), show that any product of Coxeter generators can be written without a squared term anywhere.

Tutorial 4 – The Brauer algebra



What is the Purpose of Tutorial 4?

- To learn the definition of a Brauer diagram, and how it differs from a permutation diagram
- To be able to multiply Brauer diagrams together

A new type of diagram

All the diagrams we have seen up to this point have been permutation diagrams, and the lines in these must go from top to bottom. What we will do now is a common thing in mathematics: take an established rule and tweak it slightly to see if we can find anything interesting. We will no longer insist that lines go from top to bottom, however we still need each node to be connected to precisely one other. Diagrams such as these are called "Brauer diagrams", after Richard Brauer (1901-77).

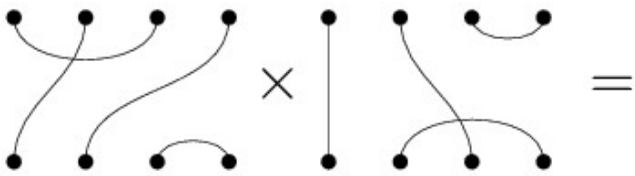
Discuss

Compare Brauer diagrams to permutation diagrams. Are there more of them or fewer? How could we count them?

Multiplication

Think about how we might multiply Brauer diagrams. Remember that since they are based on permutation diagrams, the method should be similar.

Try multiplying the following two diagrams together. What do you notice?



What to do about loops?

When we multiply two whole numbers together, we end up with another whole number. When we multiply two permutations together we end up with another permutation. Therefore when we multiply two Brauer diagrams together we want to end up with another Brauer diagram. We need to do something about these loops, since our original definition didn't include them.

Think, pair, share

Think about how we could solve this problem of the appearing loops, then discuss it with a partner. What might you need to check to make sure your solution works?

Fixing the loop problem – solution 1

One way to solve the problem of the appearing loops is to

This solution is good because

However the drawbacks are

Fixing the loop problem – solution 2

One way to solve the problem of the appearing loops is to

This solution is good because

However the drawbacks are

Fixing the loop problem – solution 3

Another way to solve the problem of the appearing loops is to

This solution is good because

However the drawbacks are

The fix

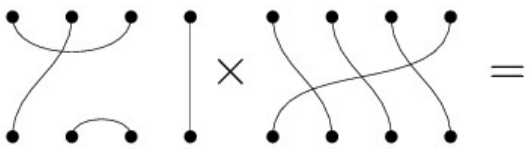
We will adopt Solution _____ because

Homework 4

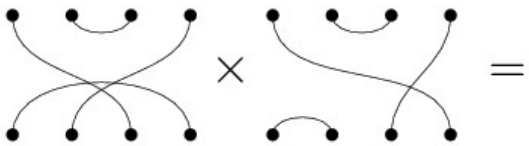
This homework will help you realise the differences and similarities between Brauer diagrams and permutation diagrams, and proved examples of the multiplication of Brauer diagrams.

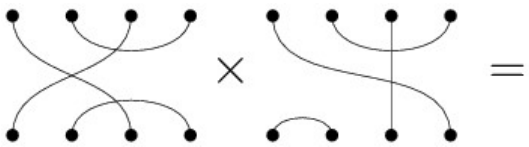
1) Write down all the Brauer diagrams with two rows of 3 nodes. (Hint: there are 15 of them)

2) Calculate the following products of Brauer diagrams

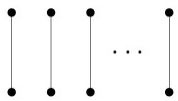
a) 

b)

c) 

d) 

3) An element x is called an "idempotent" if $x^2 = x$. The Brauer diagram



is an idempotent, find another Brauer diagram that is an idempotent and show why.

4) In a Brauer diagram, lines that join nodes on the same side are called “horizontal arcs”. Show that any Brauer diagram has the same number of horizontal arcs on the top and the bottom.

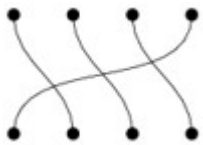
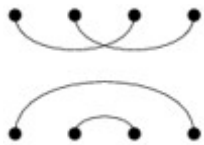
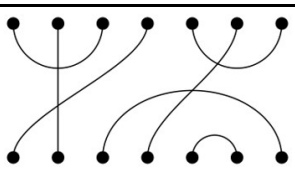
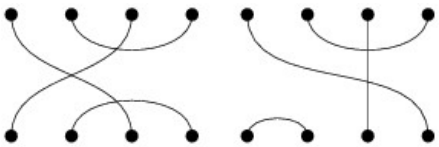
Tutorial 5 – Diving deeper into the Brauer algebra

What is the Purpose of Tutorial 5?

- To understand the "propagating number" of a Brauer diagram
- To learn how the Brauer algebra is "filtered" by propagating number

Propagating number

In Homework 4 we saw the definition of horizontal arc. The other type of line in a Brauer diagram is one that goes from top to bottom. We call these "propagating arcs". The number of propagating arcs in a given diagram is called the "propagating number" of that diagram.

Diagram	Propagating number
	
	
	
	

Parity

You may have noticed a pattern between the propagating number of a diagram and the number of nodes in each row. This is no coincidence. In fact we can prove the following two theorems:

Theorem 1) The propagating number of a Brauer diagram is even if the number of nodes in each row is even.

Theorem 2) The number of nodes in each row of a Brauer diagram is even if the propagating number is even.

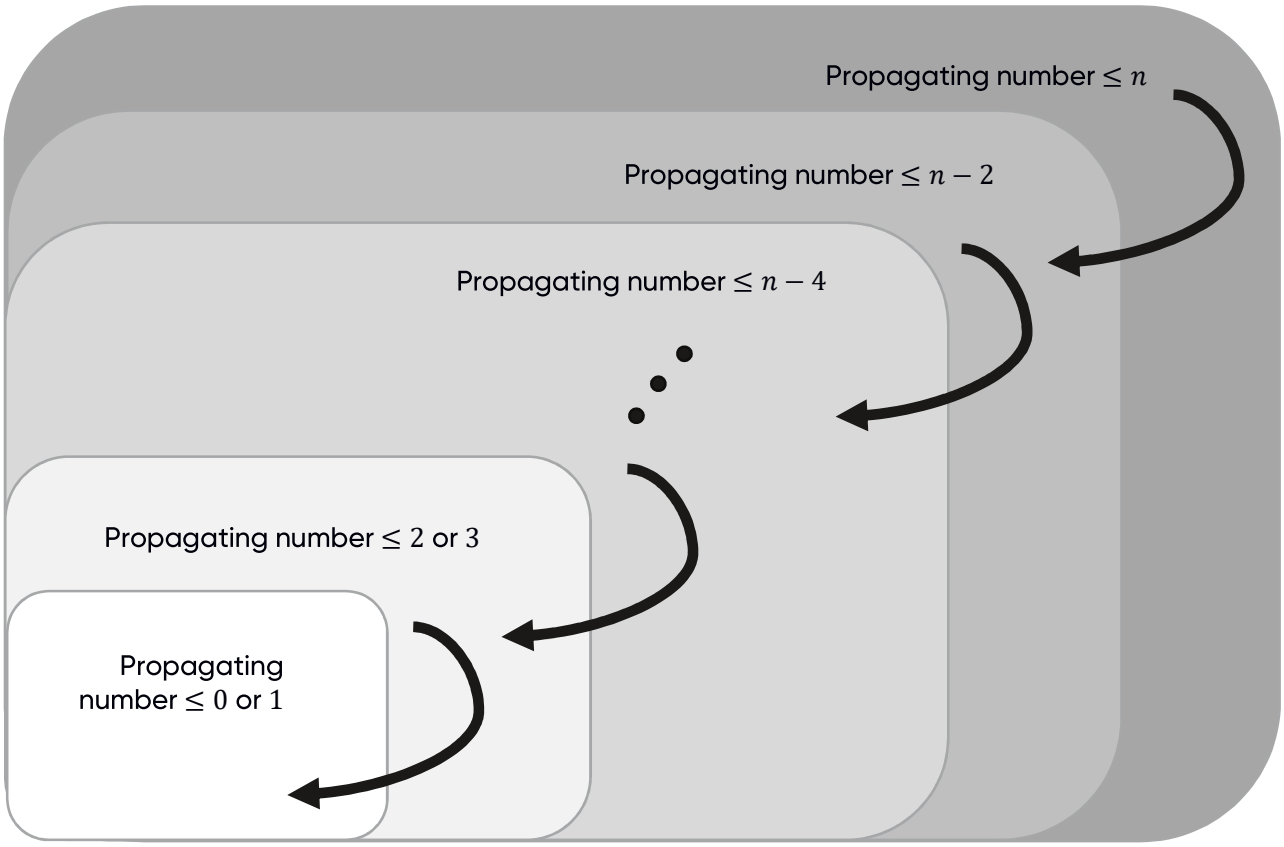
Activity

Draw a few Brauer diagrams with two rows of 5 nodes. Multiply some of them with those drawn by the person next to you. What do you notice about the propagating number of the product compared to the originals? Can you see why this happens?

Filtering the Brauer algebra

When we multiply two Brauer diagrams the propagating number of the result is

This means that we can categorise the Brauer diagrams into layers:



We say that the Brauer algebra is "filtered" by propagating number, meaning it can be arranged in these nested layers, and once you are in one layer you cannot go back up to a larger one.

Think, pair, share

Why might a filtration help mathematicians understand a mathematical structure?

Revision notes

Final Assignment – Studying a diagram algebra

Part A

This part of the assignment is checking your "hands-on" knowledge of a diagram algebra.

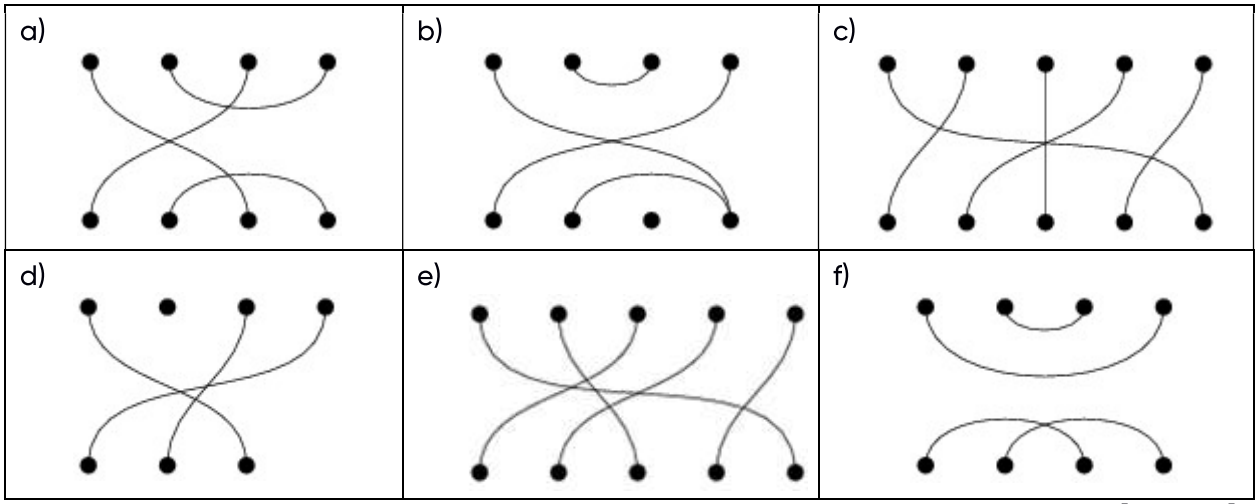
Question 1) Look at the following Cayley table and use it to calculate the products below.

↓ × →	a	b	c	d	e
a	a	b	c	d	e
b	b	0	0	b	c
c	c	b	c	0	0
d	d	0	0	d	e
e	e	d	e	0	0

a) a^2bd
b) eba
c) c^2abd
d) d^3ec
e) $b^6c^2a^4$

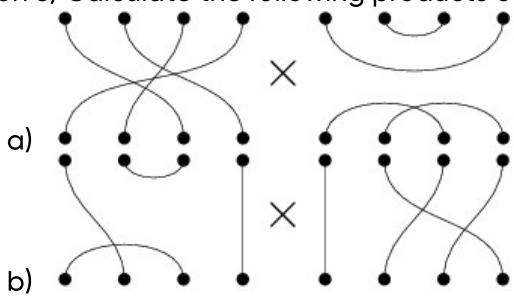
[5 marks]

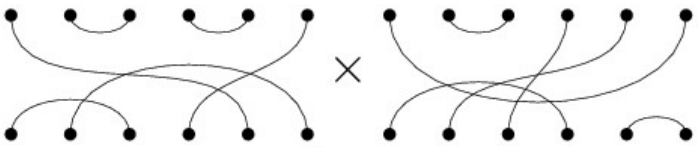
Question 2) For each of the following diagrams, label if they are
A: Permutation diagrams
B: Brauer diagrams
C: Both
D: Neither

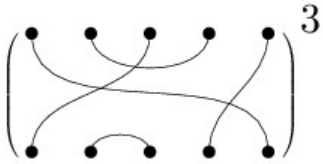


[6 marks]

Question 3) Calculate the following products of diagrams.

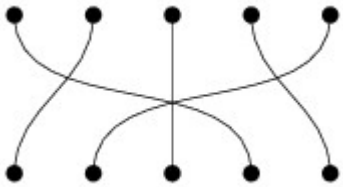


c) 

d) 

[4 marks]

Question 4) Write the following diagram as the product of Coxeter generators.



[5 marks]

Question 5) Draw the Cayley table for Brauer diagrams with two rows of 2 nodes.

[5 marks]

Part B

In this part of the assignment you will be writing proofs much like a researcher would. You may want to use the answer template on the following page to present your work like a research paper. Feel free to adapt this to make it more personal, or add in more detail. As long as you include the information from the questions, I'll be happy to check anything else you want to include!

Question 1)

a) What are the two steps in multiplying Brauer diagrams together called?

[2 marks]

b) Describe how we multiply two Brauer diagrams together. You may use pictures to illustrate the process.

[4 marks]

Question 2) Consider the "minimal generating set" of permutations on 5 objects:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}.$$

Show that any product of these generators can be written without using a power of 5 or above.

[5 marks]

Question 3) Consider the following set of permutations:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}.$$

Show that this is a generating set for permutations on 4 objects.
(Hint: Use diagram notation and show that each of the Coxeter generators can be made from these.)

[6 marks]

Question 4) Prove that when we multiply two Brauer diagrams together, the propagating number of the result cannot be greater than the propagating numbers of the originals.
(Hint: Look at what happens to the horizontal arcs on the top of the left multiplicand and the bottom of the right multiplicand.)

[8 marks]

Studying a diagram algebra

[NAME]

A diagram algebra is a set of pictures that we can "multiply" together in order to come up with a new picture.

In order to multiply diagrams in a diagram algebra we perform two steps, the first is called ___ and the second is ___. The way we do this is ___.

Instead of considering all diagrams, we can take a few (called generators) which multiply together to make all the others. When looking at permutations on 5 objects, the two generators

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

can be combined to make all others. In fact we can do this without using a power of 5 or more, as shown below. ___

We may want to use different generating sets to prove different things. Apart from the usual Coxeter generators, the following is a generating set for permutations on 4 objects:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}.$$

The proof of this is as follows. ___

Using diagrams allows us to define things like propagating number, and these give us ways of splitting up a diagram algebra into layers.

Theorem The propagating number of the product of two Brauer diagrams is less than or equal to the propagating number of both of the original diagrams.

Proof ___

Tutorial 6 – Feedback

What is the Purpose of Tutorial 6?

- To receive feedback on final assignments.
- To share examples of best practice with the other pupils in your group.
- To write targets for improvement in school lessons.
- To reflect on the programme including what was enjoyed and what was challenging.

Final assignment feedback

What I did well…	What I could have improved on…
•	•
•	•
•	•

My target for future work is…

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?
•   •   •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

Appendix 1 – Referencing correctly

When you get to university, you will need to include references in the assignments that you write, so we would like you to start getting into the habit of referencing in your Brilliant Club assignment. This is really important, because it will help you to avoid plagiarism. Plagiarism is when you take someone else's work or ideas and pass them off as your own. Whether plagiarism is deliberate or accidental, the consequences can be severe. In order to avoid losing marks in your final assignment, or even failing, you must be careful to reference your sources correctly.

What is a reference?

A reference is just a note in your assignment which says if you have referred to or been influenced by another source such as book, website or article. For example, if you use the internet to research a particular subject, and you want to include a specific piece of information from this website, you will need to reference it.

Why should I reference?

Referencing is important in your work for the following reasons:

- It gives credit to the authors of any sources you have referred to or been influenced by.
- It supports the arguments you make in your assignments.
- It demonstrates the variety of sources you have used.
- It helps to prevent you losing marks, or failing, due to plagiarism.

When should I use a reference?

You should use a reference when you:

- Quote directly from another source.
- Summarise or rephrase another piece of work.
- Include a specific statistic or fact from a source.

How do I reference?

There are a number of different ways of referencing, and these often vary depending on what subject you are studying. The most important to thing is to be consistent. This means that you need to stick to the same system throughout your whole assignment. Here is a basic system of referencing that you can use, which consists of the following two parts:

1. **A marker in your assignment**: After you have used a reference in your assignment (you have read something and included it in your work as a quote, or re-written it your own words) you should mark this is in your text with a number, e.g. [1]. The next time you use a reference you should use the next number, e.g. [2].

2. **Bibliography**: This is just a list of the references you have used in your assignment. In the bibliography, you list your references by the numbers you have used, and include as much information as you have about the reference. The list below gives what should be included for different sources.

    a. **Websites** – Author (if possible), title of the web page, website address, [date you accessed it, in square brackets].

    E.g. Dan Snow, 'How did so many soldiers survive the trenches?', http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [11 July 2014].

    b. **Books** – Author, date published, title of book (in italics), pages where the information came from.

    E.g. S. Dubner and S. Levitt, (2006) *Freakonomics*, 7-9.

    c. **Articles** – Author, 'title of the article' (with quotation marks), *where the article comes from* (newspaper, journal etc.), date of the article.

    E.g. Maev Kennedy, 'The lights to go out across the UK to mark First World War's centenary', Guardian, 10 July 2014.

Appendix 2 – Using the VLE

VLE username	
VLE password	

Please remember the following key details...

- You are able log into the VLE either through the link on our website (**www.thebrilliantclub.org**) or going directly to the VLE site at (https://portal.thebrilliantclub.org/sign-in).
- Please update your profile with your full name and email address- this will allow you to retrieve forgotten passwords or usernames
- If you forget your log-in details you can request them to be emailed to you by clicking the link on the VLE home page. (If you are still having problems you can email: **schools@thebrilliantclub.org**)

What is the VLE?

The VLE is a virtual learning environment for all pupils on Uni Pathways it is used for:

- messaging your tutor
- submitting homework
- submitting your final assignment
- accessing resources for your tutorials
- finding out more information about university and careers

How should I use the VLE?

The VLE is a professional academic environment in which pupils are able to message their PhD Tutor. Here are a few things to consider:

- Ensure you keep a professional tone in the messages you send to your tutors.
- Ensure you always reply to your tutors in a timely manner.
- Thank your tutor for the effort they are putting in to give you your feedback etc.
- Submit all homework to your tutor on time.

IMPORTANT: Final assignment

- When you submit your final assignment, please remember that you need to do so through the **'My Activities'** tab and not as an attachment to a message.

Notes

Notes

Notes

Multiplication in a world without numbers

Key Stage 4 Programme

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