Researchers in Schools

Metric spaces: measuring how far apart in different ways

Key Stage 3 Programme

Pupil Name

Coursebook Designed by Kelvin Cheung

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Timetable and Assignment Submission

Timetable – Tutorials

Tutorial	Date	Time	Location
1			
2			
3			
4			
5			
6 (Feedback)			
7 (Feedback)			

Timetable – Homework Assignments

Homework Assignment	Description	Due Date
Tutorial 1		
Tutorial 2		
Tutorial 3		
Tutorial 4		
Tutorial 5		

Assignment Submission – Lateness and Plagiarism

Lateness	
Submission after midnight on 9th August	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Course Rationale

Your teacher's PhD work was focused on the study of stochastic partial differential equations (SPDEs). These equations are used describe physical phenomena in different fields such as physics, engineering and finance. Solutions of these equations live in special mathematical objects called metric spaces. These spaces are sets endowed with a certain notion of distance. While SPDEs are very much beyond the scope of the year 9 curriculum, the concept of metric spaces is actually very intuitive to understand. This course serves as a short introduction of these spaces and how they relate to the very world we live in.

The main goal of this course is to give secondary school students a glimpse of what university style learning is like. In mathematics, the biggest change going from school to university learning is that university students are expected write a lot of mathematical proofs, something that is rarely touched upon in school learning. In this course, students will be exposed to proof writing in a gentle pace, starting from writing a simple geometric proof of Pythagoras theorem to showing that a distance function satisfies the definition of a metric. Students will also be asked to take their own notes in later tutorials, as this is an important skill to have regardless of what course the students choose to take in university.

The contents of the 5 main tutorials will be as follows:

Tutorial 1: Rational numbers, irrational numbers, and real numbers.

Tutorial 2: Pythagoras theorem and its relation to Euclidean distances and spaces.

Tutorial 3: Different notions of distances, and examples of distances outside of Euclidean distances.

Tutorial 4: Definition of a metric space, and how to check a distance function is a metric.

Tutorial 5: Examples of metric spaces that are not defined on the Euclidean spaces.

Uni Pathways Mark Scheme 2021 – maths-based assignments

	Subject Knowledge	Critical Thinking	Written Communication
1st	The work shows a depth of knowledge and understanding of key concepts and mathematical methods, drawing on sources of evidence where appropriate. Knowledge is used to build and support highly effective mathematical arguments and explanations.	Analyses the available data/evidence and identifies all the relevant mathematical information. Identifies and critically evaluates available mathematical approaches and evidence, deciding on their credibility, strength and relative significance. Makes connections between different pieces of information and puts them together to form accurate solutions and verify answers and assumptions.	Mathematical arguments are complete, and the reasoning behind each step has been clearly communicated to the reader. The work has a coherent flow and is well structured. The writing style is appropriate; mathematical language and key mathematical terms are used accurately and effectively to support the arguments and explanations made. There are no, or very few, errors in spelling or grammar. Consistent referencing, appropriate paragraphing and use of correctly labelled tables and figures; matching the style taught in the course.
2:1	The work shows an understanding of key concepts and mathematical methods, drawing on sources of evidence where appropriate. Knowledge is used to build and support effective mathematical arguments and explanations.	Analyses the available data/evidence and identifies all the relevant mathematical information. Engages with multiple approaches and critically evaluates their utility, forming judgements and using reasoning and evidence to inform problem solving strategy. Shows some understanding of the relative value of evidence and data.	Mathematical arguments are clearly made, with few gaps in logical reasoning. The work is well-structured. The writing style is appropriate; mathematical language and key terms are used correctly. There are few errors in spelling or grammar. Mostly consistent referencing and use of tables and figures; matching the style taught in the course.
2:2	The work shows an understanding of key concepts and mathematical methods, with no major misconceptions. Beginning to apply this knowledge to build and support effective mathematical arguments and explanations.	Identifies and uses basic mathematical evidence and reasoning. Evaluates the quality of mathematical reasoning and evidence, forming judgements. Not yet showing understanding of the relative value of evidence and reasoning.	Mathematical arguments have been constructed, but there is little explanation from step to step. The work has some structure. The writing style can sometimes be informal; occasionally key terms are not used when it would be appropriate to do so.

			There are some errors in grammar and spelling, but these do not get in the way of communicating the content. Referencing has some consistency; matching the style taught in the course. Limited use of tables and figures.
3rd	Shows a developing understanding of key concepts and mathematical methods, with some misconceptions. Not yet applying this knowledge to build and support mathematical arguments and explanations.	Beginning to analysis mathematical evidence and reasoning. Describes or attempts to solve the problem without any/or little evaluation of whether the approach was reasonable or accurate.	Mathematical arguments to solve problems are poorly sequenced, with little or no explanation. The grammar, spelling, style, and structure of the work need improving in order to communicate ideas to the reader. Key terms are not always used correctly. Limited, or no use of referencing, tables and figures.

Baseline Assignment (problem-set): Pupil Feedback Report

Name of Pupil	
Name of School	
Name of RIS teacher	
Title of Assignment	

How your assignment is graded:

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-level
2:1	60-69	Performing to a good standard at A-level
2:2	50-59	Performing to an excellent standard at GCSE
3rd	40-49	Performing to a good standard at GCSE
Working towards a pass	0-39	Performing below a good standard at GCSE
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Marks

OVERALL MARK / 100		FINAL MARK / 100 (including any deductions)	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the teacher should give an explanation in this section:

Mark Breakdown and Feedback

Subject knowledge	
	mark
Critical thinking	
	mark
Written communication	
	mark

Problem Set	
Comment	*mark*

Final Assignment (problem-set): Pupil Feedback Report

Name of Pupil	
Name of School	
Name of RIS teacher	
Title of Assignment	

How your assignment is graded:

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-level
2:1	60-69	Performing to a good standard at A-level
2:2	50-59	Performing to an excellent standard at GCSE
3rd	40-49	Performing to a good standard at GCSE
Working towards a pass	0-39	Performing below a good standard at GCSE
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Marks

OVERALL MARK / 100		FINAL MARK / 100 (including any deductions)	
DEDUCTED MARKS		FINAL GRADE	

If marks have been deducted (e.g. late submission, plagiarism) the teacher should give an explanation in this section:

Mark Breakdown and Feedback

Subject knowledge	
	mark
Critical thinking	
	mark
Written communication	
	mark
Problem Set	
Comment	*mark*

Subject Vocabulary

Word	Definition	Example
Rational number	A number that can be expressed as a fraction with integer numerator and denominator.	See Tutorial 1.
Irrational number	A number that cannot be expressed as a fraction with integer numerator and denominator.	See Tutorial 1.
Real number	The set of rational and irrational numbers.	See Tutorial 1.
Theorem	A mathematical statement that has been proven to be true.	A famous theorem is the Pythagoras Theorem which gives a relation between the 3 sides of a right-angled triangle.
Proof	A logical step-by-step argument showing that something is true.	Pythagoras Theorem can be proved by rearranging geometric shapes (See Tutorial 2).
Distance function	A function taking two things x and y as inputs and a non-negative real number as an output.	See Tutorial 3.
Euclidean space	A mathematical space used to model the very world we live in.	See Tutorial 2.
Euclidean distance	"Straight-line" distance between two points (see Tutorial 2).	See Tutorial 2.
Metric	A distance function that satisfies the four conditions of a metric (see Tutorial 4).	See Tutorial 4.
Metric space	A set X together with a metric d.	See Tutorial 4.
Graph	A network comprised of vertices joined together by edges (see Tutorial 5).	See Tutorial 5.

Launch Event

The launch event is your opportunity to learn more about this course and Uni Pathways. You will watch some videos, listen to your teacher and have some discussion. Below is some room to make notes for some of the videos you will watch during the event.

Virtual Campus Tours

Notes…

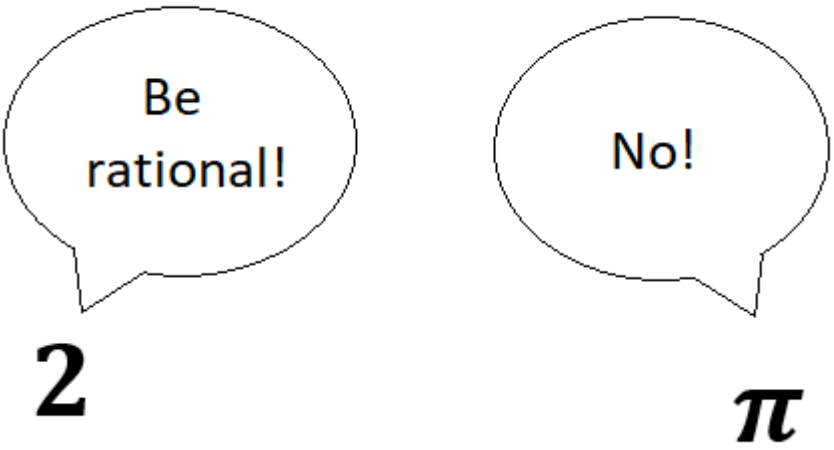
The Scholars Programme Alumni

Notes…

Student Ambassadors

Notes…

Tutorial 1 – Real numbers: how to be rational and irrational



What is the Purpose of Tutorial 1?

- To learn what rational and irrational numbers are.
- To learn how to tell whether a number is rational or not.
- To learn what surds are and how to do some basic operations with them.

Definitions

Rational numbers are numbers that can be written as a fraction where both the *numerator* and *denominator* are *integers*.

Irrational numbers are numbers that are *not rational*.

Real numbers consist of both rational numbers and irrational numbers – basically every number you have seen in your life would most likely be a real number!

Task 1 – Classify

You will be working in pairs to decide whether the following numbers are rational or not. Tick the appropriate box and for those that are rational, give a short explanation as to why they are.

	Rational?	Irrational?	If rational, please explain…
41			
$3/8$			
$\sqrt{10}$			
$\sqrt{9}$			
-9999			

Key concept – adding and multiplying surds

Surds are roots of numbers that cannot be "worked out" as a rational number, e.g.

$$\sqrt{2} \qquad \sqrt{5} \qquad \sqrt[3]{10}$$

We will be mostly working with surds that are square roots of integers.

When adding surds, we can only combine them if they are of the same kind, for example

- $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$
- $\sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$

If the surds are not the same kind, then we cannot combine them and can only leave them as they are. For example, $\sqrt{2} + \sqrt{3}$ cannot be simplified further. This is similar to combining like terms in algebra.

When multiplying surds, we simply need to multiply the numbers inside the radical sign $\sqrt{}$. For example,

$$\sqrt{2}\sqrt{3} = \sqrt{6}.$$

Try it yourself: Simplify the following expressions of surds:

1) $3\sqrt{11} + 2\sqrt{11} =$ _____

2) $\sqrt{2} + \sqrt{2} + \sqrt{3} =$ _____

3) $\sqrt{5}\sqrt{3} =$ _____

4) $\sqrt{7}\sqrt{5} =$ _____

Task 2 – Brain teaser

John is playing around with multiplying numbers together. After some experimenting, he makes the following claims:

1) The product of two rational numbers is always rational
2) The product of two irrational numbers is always irrational

You will be working in pairs to decide whether John's claims are correct. If you think a claim is correct, try to give an explanation for it. On the other hand, if you think a claim is incorrect, try to find a counterexample (i.e. an exception to the claim).

Is Claim 1) correct?

Is Claim 2) correct?

Key concept – Rationalising denominators

When dealing with fractions containing surds, it is often useful to "rationalise" the denominator. This means converting the denominator into an integer. For example, let's look at the number

$$\frac{\sqrt{2}}{\sqrt{3}}$$

For this number, we notice that if we do $\sqrt{3} \times \sqrt{3}$, we get $\sqrt{9} = 3$. Therefore, we can rationalise the denominator by multiplying it by $\frac{\sqrt{3}}{\sqrt{3}}$.

$$\frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Notice that this does not change the value of the original number, because we are simply multiplying the whole fraction by 1.

Task 3 – Rationalisation exercise

Rationalise the denominators in the following fractions:

$$\frac{\sqrt{5}}{\sqrt{2}}$$

$$\frac{1}{\sqrt{3}}$$

$$\frac{\sqrt{10}}{\sqrt{5}}$$

Imagine being asked to compare the sizes of the numbers $\frac{\sqrt{2}}{\sqrt{3}}$ and $\frac{3}{\sqrt{6}}$. It is quite difficult to see which one is bigger by just looking at it, isn't it?

Now try to rationalise these two numbers and see if you can compare them now. Which one is bigger?

Final note

Irrational numbers might seem strange and absurd at the start, indeed, that is where the word "surd" comes from! However, as it turns out, there are actually far, far more irrational numbers than rational numbers out there. After all, if you choose a random positive integer and take its square root, then you are far more likely to get back a surd than an integer. In this sense, irrational numbers can be viewed to be far more natural than the usual integers and fractions we are used to!

Tutorial 2 – Euclidean spaces and distances

Figure 2.1: Pythagoras (left) and Euclid (right), two Greek mathematicians.

What is the Purpose of Tutorial 2?

- To learn what Euclidean spaces are.
- To learn how to use Pythagoras Theorem and see a proof of it.
- To learn what Euclidean distances are and how Pythagoras Theorem is related to that.

The real line - $\mathbb{R}$

In the last tutorial, we learnt that the **real numbers** consist of all the rational and irrational numbers. Altogether, these real numbers form an infinite number line known as the **real line**. We often use the symbol $\mathbb{R}$ to represent the real line. $\mathbb{R}$ is also known as the **one-dimensional Euclidean space**.

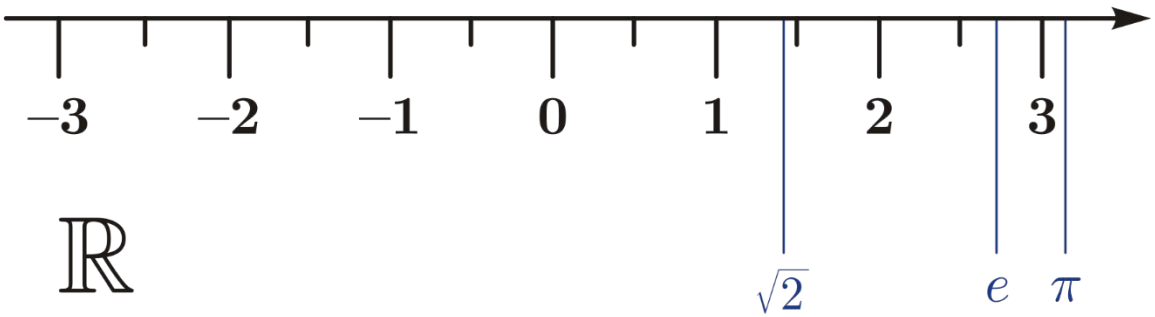


Figure 2.2: a short section of the real line $\mathbb{R}$ from Wikipedia.

Absolute value sign on $\mathbb{R}$

For a real number x, we often use the **absolute value** symbol $|x|$ to represent the positive value of x. This is also known as the **modulus** sign.

<u>Examples:</u>

- $|-3| = 3$
- $|5| = 5$
- $|4 - 1| = 3$
- $|1 - 4| = 3$

Try it yourself: Work out the following absolute values:

5) $|-9| =$ _____

6) $|8| =$ _____

7) $|3-6| =$ _____

8) $|0| =$ _____

Euclidean distance on $\mathbb{R}$

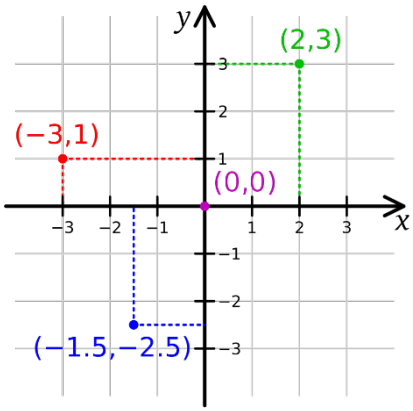
The **Euclidean distance** between two points can simply be thought of as the "straight-line distance" between them. Calculating the Euclidean distance on $\mathbb{R}$ is very straightforward: you just need to take the difference between the two points!

Example:
The Euclidean distance between 3 and 2 is 1.

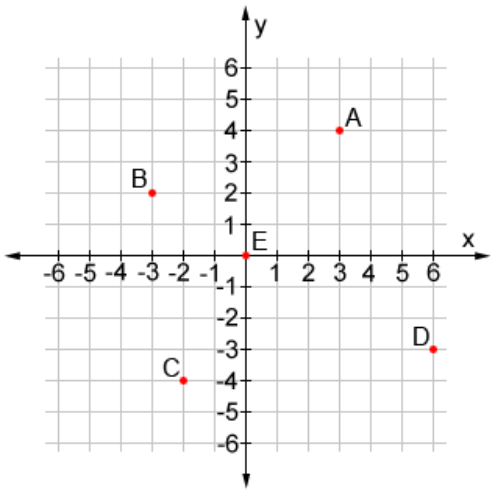
We often use the absolute value symbol $|\ldots|$ to represent Euclidean distances, so for example, $|3-2|$ represents the Euclidean distance between 3 and 2.

The 2D plane - $\mathbb{R}^2$

The two dimensional Euclidean plane (also known as Cartesian plane) is used to represent a flat space, like the floor you are standing on, or a piece of paper. There are two axes to measure where the points are on the plane: the horizontal x-axis and the vertical y-axis. We use the symbol $\mathbb{R}^2$ to represent this plane.



Try it yourself: Label the points in the following grid representing $\mathbb{R}^2$:



$A = ($ _____ , _____ $)$

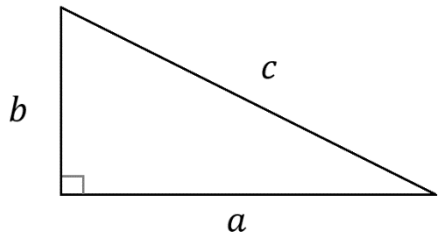
$B = ($ _____ , _____ $)$

$C = ($ _____ , _____ $)$

$D = ($ _____ , _____ $)$

Key concept – Pythagoras Theorem

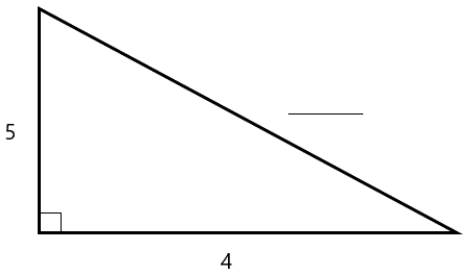
Pythagoras Theorem states that in any right-angled triangle, the lengths of the sides are related in the following way:

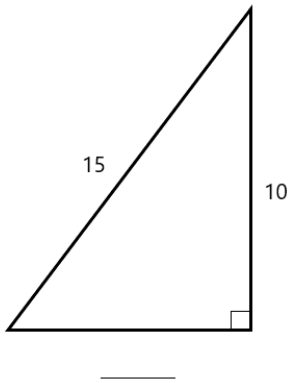


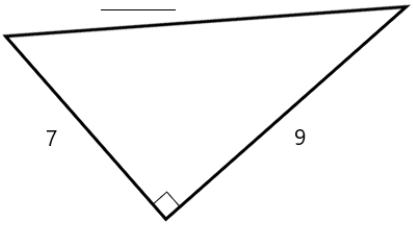
$$\boxed{a^2 + b^2 = c^2}$$

Task 1 – Triangles

Use a calculator to work out the missing sides, rounding to 3 significant figures if necessary:



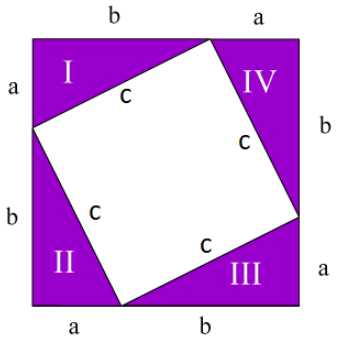




Task 2 – Proof of Pythagoras Theorem

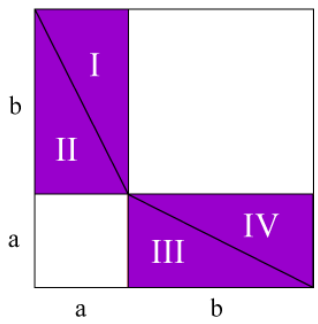
For this task, you will be following the guide below to write a proof of Pythagoras Theorem! (This proof is given by Pythagoras himself)

Consider arranging four of the same right-angled triangles (labelled I, II, III and IV) in the following way:



What is the area of the white square enclosed by the triangles?

_____



What is the total area of the white spaces now?

_____

Explain how this proves Pythagoras Theorem:

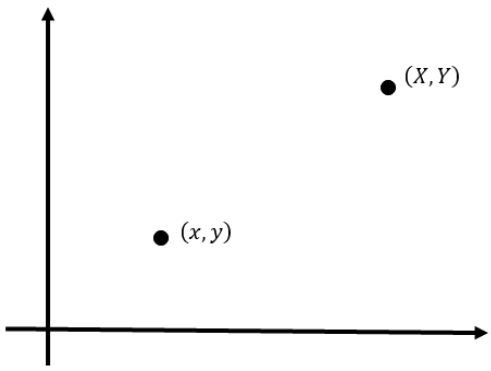
_____

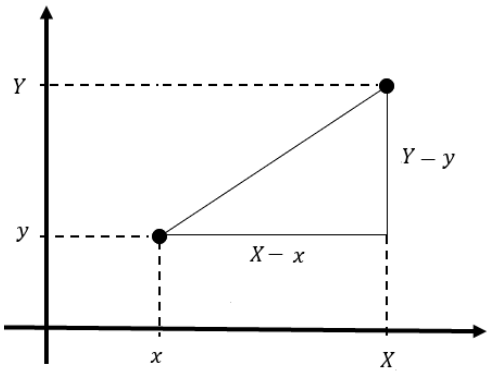
_____

_____

Euclidean distance on $\mathbb{R}^2$

We are now ready to see what **Euclidean distance** in $\mathbb{R}^2$ looks like. Consider two points (x, y) and (X, Y) in $\mathbb{R}^2$:





We can draw a right-angled triangle on the two points as seen on the right picture. By Pythagoras Theorem, the Euclidean distance D is given by

$$D^2 = (X - x)^2 + (Y - y)^2$$

$$\boxed{D = \sqrt{(X - x)^2 + (Y - y)^2}}$$

Task 3 – Distances on $\mathbb{R}^2$

Use a calculator to work out the Euclidean distances of the following pairs of points on $\mathbb{R}^2$, rounding numbers to 3 significant figures if necessary:

$(3,2)$ and $(4,5)$ _____

$(10,3)$ and $(1,7)$ _____

$(-1,-5)$ and $(2,0)$ _____

Homework

Use a calculator to work out the Euclidean distances of the following pairs of points on $\mathbb{R}^2$, rounding numbers to 3 significant figures if necessary:

$(10,10)$ and $(0,0)$ _____

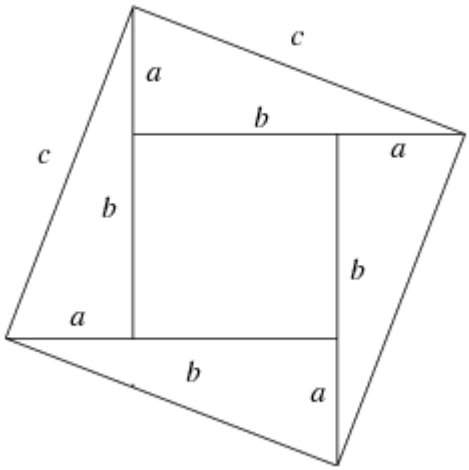
$(2,8)$ and $(2,8)$ _____

$(-1,-1)$ and $(1,1)$ _____

Another proof of Pythagoras Theorem

While Pythagoras was the first to discover his theorem, many different proofs were discovered independently around the world. In this exercise, we look at a proof by Bhāskara, an Indian mathematician in the 12th century.

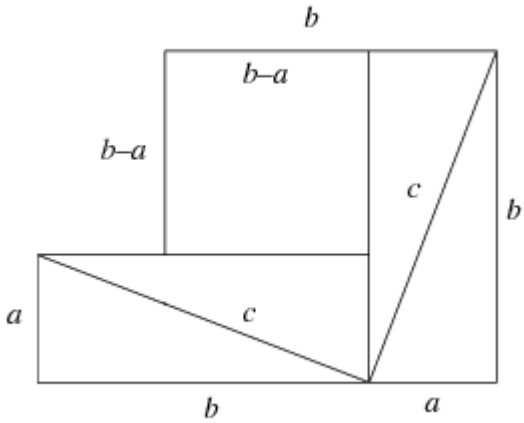
Consider arranging four of the same right-angled triangles in the following way:



What is the area of the big square?

_____

We now rearrange the shapes in the following way:



What is the total area of the whole shape now?
(HINT: try to split the diagram into two squares)

_____

Explain how this proves Pythagoras Theorem:

_____

_____

_____

Tutorial 3 – Different notions of distances

What is the Purpose of Tutorial 3?

- To appreciate the need to have different types of distances.
- To learn what Manhattan distance on $\mathbb{R}^2$ is.
- To learn what Los Angeles distance on $\mathbb{R}^2$ is.

Task 1 – Is Euclidean distance good for this situation?

Take a look at the photo of the city of Manhattan on the right.

Is the Euclidean distance good a good indicator for how long it takes to travel from one point to the other?

Discuss with the person next to you for 2 minutes and summarise your thoughts below:

_____

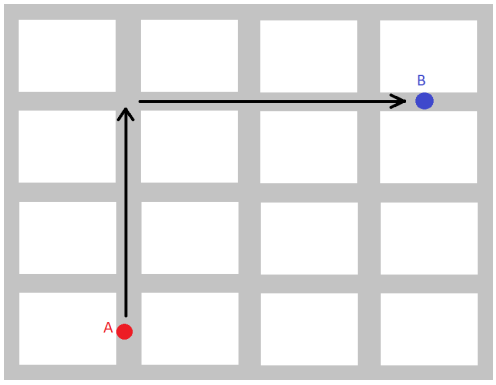
_____

_____

_____

Manhattan distance on $\mathbb{R}^2$

Consider the following simple model for Manhattan city, where the white rectangles represent buildings while the gray lines represent roads. To travel from A to B, we would have to travel along the roads. Therefore, it makes more sense to measure the distance of the roads travelled than to measure the Euclidean distance between the two points.



This gives rise to the **Manhattan distance** on $\mathbb{R}^2$.

The **Manhattan distance** between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in $\mathbb{R}^2$ is given by

$$d_M(A, B) = |x_1 - x_2| + |y_1 - y_2|$$

Note: the Manhattan distance is also known as the *taxicab* distance or the L^1 distance.

<u>Examples:</u>

- $A = (9, 3)$ and $B = (4, 2)$. Then $d_M(A, B) = |9 - 3| + |3 - 2| = 6 + 1 = 7$.
- $A = (2, 0)$ and $B = (3, 5)$. Then $d_M(A, B) = |2 - 3| + |0 - 5| = 1 + 5 = 6$.

Try it yourself: Work out the Manhattan distance of A and B on $\mathbb{R}^2$:

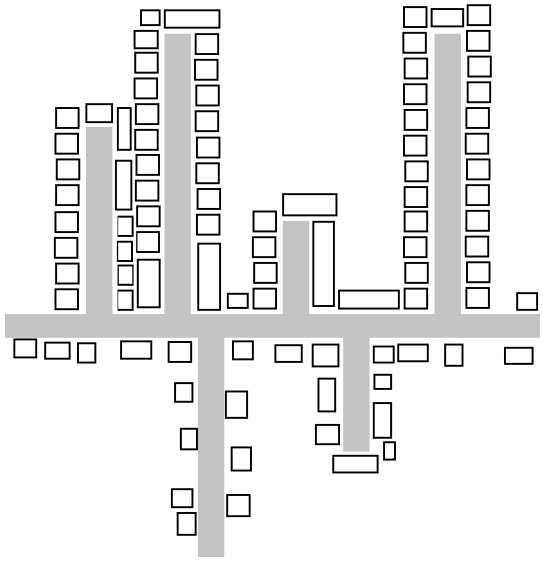
9) $A = (5, 4)$ and $B = (3, 1)$
Then $d_M(A, B) =$ _____

10) $A = (0, 0)$ and $B = (5, 5)$
Then $d_M(A, B) =$ _____

11) $A = (2, 7)$ and $B = (-1, -2)$
Then $d_M(A, B) =$ _____

12) $A = (-3, 6)$ and $B = (4, -8)$
Then $d_M(A, B) =$ _____

Task 2 – City with a highway



Imagine you are living in a city that looks like the picture on the right, where the gray parts represent roads and the black rectangles represent buildings. To get anywhere in the city that is not within your road, you need to travel through the central highway.

Discuss with the person next to you and try to answer the following questions.

1) Why is the Euclidean distance a bad way to measure the distance travelled in this city?

   _____

   _____

   _____

2) Why is the Manhattan distance not a good way to measure distance travelled in this city either?

   _____

   _____

3) **Challenge:** By assuming that the city is modelled on the $\mathbb{R}^2$ plane, with the middle highway being the x-axis, try to come up with a formula that is useful to measure distance travelled in this city:

   _____

   _____

   _____

   _____

Los Angeles distance on $\mathbb{R}^2$

The notion of distance hinted from question 3) above is known as the Los Angeles distance (this comes from the fact that there is a freeway in Los Angeles which connects to many parts of the city). The Los Angeles distance between two points A and B is given by

$$d_{LA}(A, B) = _____$$

<u>Examples:</u>

- $A = (9, 3)$ and $B = (4, 2)$. Then $d_{LA}(A, B) =$ _____
- $A = (2, 0)$ and $B = (3, 5)$. Then $d_{LA}(A, B) =$ _____

Try it yourself: Work out the Los Angeles distance of A and B on $\mathbb{R}^2$:

1) $A = (5, 4)$ and $B = (3, 1)$
   Then $d_{LA}(A, B) =$ _____

2) $A = (0, 0)$ and $B = (5, 5)$
   Then $d_{LA}(A, B) =$ _____

3) $A = (2, 7)$ and $B = (-1, -2)$
   Then $d_{LA}(A, B) =$ _____

4) $A = (-3, 6)$ and $B = (4, -8)$
   Then $d_{LA}(A, B) =$ _____

Homework

Work out the Manhattan distance and the Los Angeles distance of A and B on $\mathbb{R}^2$ for the following:

1) $A = (0, 0)$ and $B = (1, 1)$
   $d_M(A, B) =$ _____
   $d_{LA}(A, B) =$ _____

2) $A = (6, -1)$ and $B = (3, 9)$
   $d_M(A, B) =$ _____
   $d_{LA}(A, B) =$ _____

3) $A = (3, 2)$ and $B = (-5, -1)$
   $d_M(A, B) =$ _____
   $d_{LA}(A, B) =$ _____

4) $A = (-7, 2)$ and $B = (7, 1)$
   $d_M(A, B) =$ _____
   $d_{LA}(A, B) =$ _____

Tutorial 4 – Metric space

What is the Purpose of Tutorial 4?

- To learn what triangle inequality is, and to prove it.
- To learn what a metric space is.
- To learn how to prove that a notion of distance forms a metric.

Triangle inequality

The triangle inequality is the following relation that holds true for *any* real numbers a and b.

$$|a+b| \leq |a| + |b|$$

Try it yourself:
Verify that the triangle inequality works for the following choices of a and b.

$a = 3$	$b = 3$	$\lvert a+b\rvert =$_____	$\lvert a\rvert + \lvert b\rvert =$_____	Is $\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$? _____
$a = -3$	$b = -3$	$\lvert a+b\rvert =$_____	$\lvert a\rvert + \lvert b\rvert =$_____	Is $\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$? _____
$a = -3$	$b = 3$	$\lvert a+b\rvert =$_____	$\lvert a\rvert + \lvert b\rvert =$_____	Is $\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$? _____
$a = 5$	$b = -2$	$\lvert a+b\rvert =$_____	$\lvert a\rvert + \lvert b\rvert =$_____	Is $\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$? _____

Take two minutes and discuss with the person next to you. In one or two sentences, why do you think the triangle inequality is true for any numbers?

_____

_____

Task 1 – Proving the triangle inequality

In this task, you will work through the following questions which will eventually lead up to a proof of the triangle inequality.

Let a and b be *any* real numbers. We split the proof into three cases:

1) If $a \geq 0$ and $b \geq 0$, rewrite $|a+b|$ and $|a| + |b|$ without the absolute value signs:

$$|a+b| = _____$$

$$|a| + |b| = _____$$

Therefore, (fill in the blank with either <, = or >)

$$|a+b| _____ |a| + |b|$$

2) If $a < 0$ and $b < 0$, rewrite $|a+b|$ and $|a| + |b|$ without the absolute value signs:

$$|a+b| = _____$$

$$|a| + |b| = _____$$

Therefore, (fill in the blank with either <, = or >)

$$|a+b| _____ |a| + |b|$$

3) If $a < 0$ and $b \geq 0$, then we consider two subcases:
   i) If $|a| \leq |b|$, then $|a + b| \leq |b|$. Why? (HINT: it might help to draw a number line)

   _____

   ii) If $|a| > |b|$, then $|a + b| \leq |a|$. Why? (HINT: it might help to draw a number line)

   _____

   In either case, we conclude $|a + b| \leq |a| + |b|$.

Metric space - unifying the notions of distances

In previous tutorials, we learnt about different types of distances. While they all quite different, they do share some similarities.

The four conditions that a distance function should satisfy are:

1) _____

2) _____

3) _____

4) _____

If the above 4 conditions are satisfied for a distance function d in a set X, then we can call X a **metric space** with **metric** d.

As it turns out, all the distances we studied previously do indeed satisfy the above 4 conditions.

Euclidean distance on $\mathbb{R}$ is a metric

Recall that the Euclidean distance between two points x, y on $\mathbb{R}$ is given by

$$d_E(x, y) = |x - y|$$

Let's check that d_E on $\mathbb{R}$ satisfies the conditions of a metric space.

Condition 1:

_____

Condition 2:

_____

Condition 3:

_____

Condition 4:

_____

Task 2 – Show that Manhattan distance on $\mathbb{R}^2$ is a metric

Recall that the Manhattan distance between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ on $\mathbb{R}^2$ is given by

$$d_M(A, B) = |x_1 - y_1| + |x_2 - y_2|$$

Follow the example above and show that d_M on $\mathbb{R}^2$ satisfies the conditions of a metric space.

Condition 1:

Condition 2:

Condition 3:

Condition 4:

Homework

Recall that the Los Angeles distance between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ on $\mathbb{R}^2$ is given by

$$d_{LA}(A, B) = |x_1| + |x_1 - x_2| + |y_2|$$

Follow the examples from the lesson and show that d_{LA} on $\mathbb{R}^2$ satisfies the conditions of a metric space.

Condition 1:

Condition 2:

Condition 3:

Condition 4:

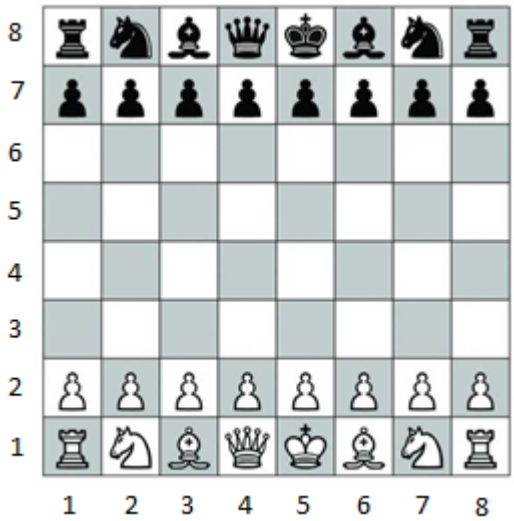
Tutorial 5 – Metric spaces outside $\mathbb{R}$ and $\mathbb{R}^2$

What is the Purpose of Tutorial 5?

- To learn what the Chebyshev distance is on a chessboard.
- To learn what a graph is and what graph distance is.
- To practice note taking skills which is crucial in university learning environment.

In previous tutorials, we have seen a good number of metric spaces where the set X is either $\mathbb{R}$ and $\mathbb{R}^2$. The purpose of this tutorial is to showcase some metric spaces that are not on these sets.

Chebyshev distance on a chessboard



A chessboard is an 8×8 grid where a chess game is played. Professional chess players often use the Chebyshev distance on a chessboard.

If $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are two points on the chessboard, then the Chebyshev distance between A and B is given by

$$d_C = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

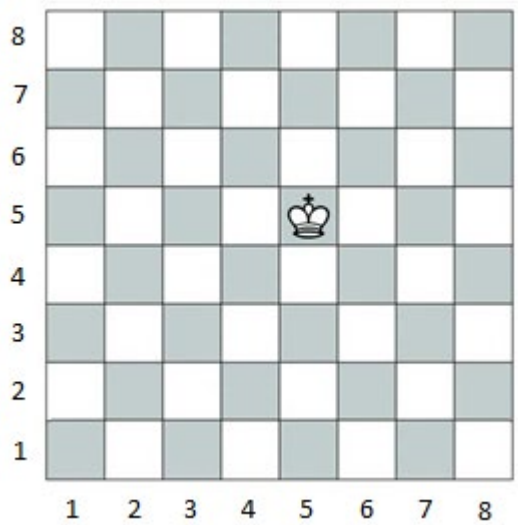
Examples

1) If $A = (1,3)$ and $B = (2,6)$, then
$$d_C = \max\{|2-1|, |6-2|\} = \max\{1,4\} = 3$$

2) If $A = (5,1)$ and $B = (1,2)$, then
$$d_C = \max\{|5-1|, |1-2|\} = \max\{4,1\} = 4$$

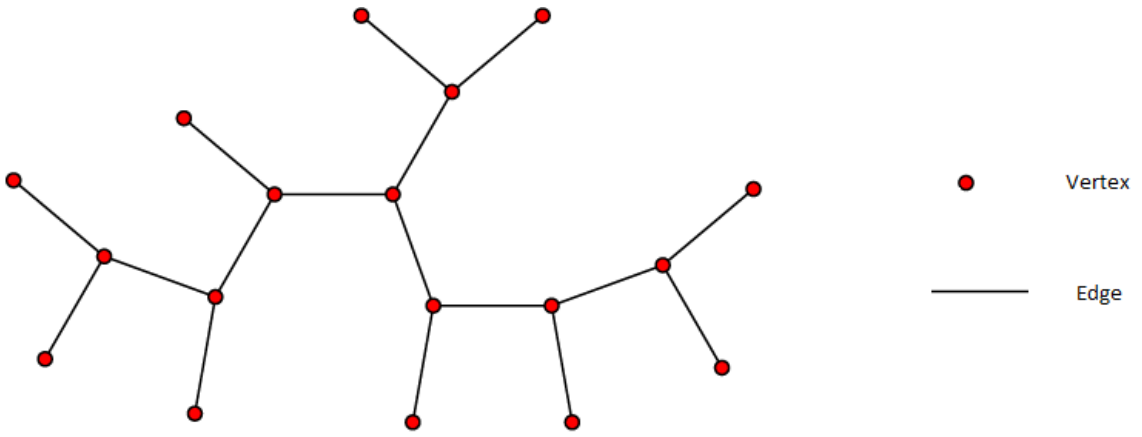
Task 1 – Visualising Chebyshev distance

On the following chessboard, there is a king piece on (5,5). On each of the squares of the board, write down the Chebyshev distance between that square and the king. What do you see?



Graphs

In mathematics and computer science, a graph is a diagram composed of points called vertices joined up by edges (not to be confused by a graph of a curve).



Notes:

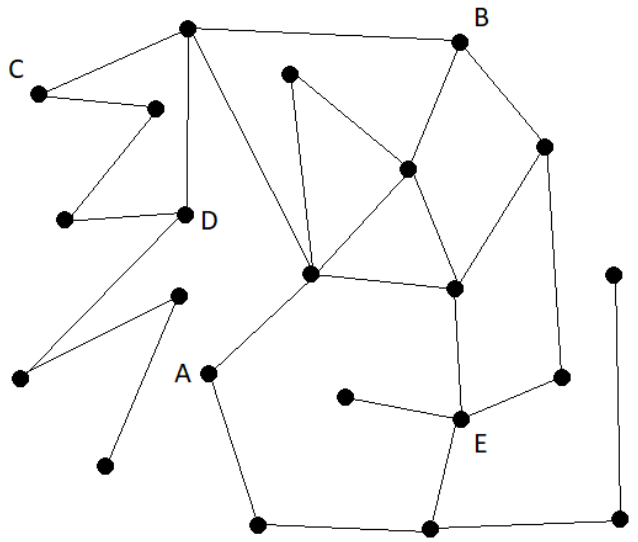
Graph distance

The graph distance $d_G(a, b)$ between two points A and B in a graph G is given by

_____

Task 2 – Working with graph distance

Work out the graph distances for the graph below:



$d_G(A, B) =$ _____

$d_G(C, D) =$ _____

$d_G(D, E) =$ _____

$d_G(A, E) =$ _____

$d_G(C, E) =$ _____

$d_G(B, D) =$ _____

Give two vertices that are connected by more than one path of the same length:

_____

Find the two vertices (among the unlabelled ones) that have the greatest graph distance between them, label them P and Q in the above graph. What is the graph distance between P and Q?

$$d_G(P, Q) = _____$$

Challenge – Show that the graph distance is a metric

If you have finished all the above tasks. Use the space below and try to write a proof showing that the graph distance d_G on a graph G satisfy all four conditions of a metric space.

(HINT: As usual, Condition 4 is the one that is the most difficult to prove. Try to argue by contradiction: assume that Condition 4 is false, can you find a contradiction somehow?)

_____

_____

_____

_____

Final assignment

The final assignment consists of a problem set related to the content of the 5 tutorials you have attended throughout this course. The success criteria is the same as in the baseline assignment: the solutions to the problems should be clearly written, and all your working clearly laid out.

1)
   a) Explain the meaning of the terms "rational number" and "irrational number". Give two examples of each.
   b) Simplify the following fractions of surds:
      i) $\frac{\sqrt{2}}{\sqrt{10}}$
      ii) $\frac{\sqrt{7}}{\sqrt{6}}$
      iii) $\frac{2}{\sqrt{2}}$
   c) Decide whether the following numbers are rational or irrational. Give an explanation for those which are rational.
      i) $\sqrt{2}$
      ii) 94312
      iii) $-\pi$
      iv) $\sqrt{2} \times \sqrt{2}$

2)
   a) Calculate the Manhattan distance between the following pairs of points on $\mathbb{R}^2$.
      i) $(3, 10)$ and $(1, 34)$
      ii) $(-6, 5)$ and $(-2, -9)$
      iii) $(16, -6)$ and $(-5, 9)$
   b) Draw a graph with 6 vertices A, B, C, D, E, F with the following properties:
      i) $d_G(A, E) = 3$
      ii) $d_G(A, C) = 2$
      iii) $d_G(C, E) = 3$
      iv) $d_G(F, A) = 3$

      Look at the graph you have drawn? What must $d_G(B, D)$ be?

3) In a fantasy story, the world is split into two realms: the Overworld andthe Nether. These two realms are modelled using the Euclidean plane $\mathbb{R}^2$.

   a) Alex and Steve are in the Overworld at the coordinates $(1, 10)$ and $(4, 15)$ respectively. What is the Euclidean distance between them? Write your answer in 3 significant figures.

   In order to traverse between the Overworld and the Nether, Alex and Steve need to construct portals. The portal mechanism works as follows: if a portal P is constructed on a point $A = (x, y)$ in the Overworld, then P will search for existing portals in the Nether that are at a distance of at most 16 from the point $A' = \left(\frac{x}{8}, \frac{y}{8}\right)$, and link with the portal P' closest to A' . If multiple closest portals exist in the Nether, then P will link with a random one among the closest portals.

   b) Alex built a portal in the overworld at the point $A = (10, 30)$. If there are existing portals in the Nether at the coordinates $(1,6)$, $(15,30)$ and $(7,9)$, which of these coordinates will Alex link to?
   c) The portal mechanics essentially means that "every step taken in the Nether is 8 steps in the Overworld". Steve wants to exploit this to create a "fast travel" system through the Nether. He has built two portals in the Overworld: one near his home at $H = (0,0)$ and one near his swimming pool at $(480,480)$. He now needs to build two portals H' and S' in the Nether to link to H and S respectively, where should he build these if he wants to minimise the distance needed to travel from H' to S'?

Tutorial 6 – Feedback tutorial

What is the Purpose of Tutorial 6?

- To receive feedback on your final assignment
- To respond to the feedback from your Uni Pathways teacher
- To write targets for improvement on your final assignment

Final assignment feedback from your Uni Pathways Teacher

(Remember to look at the mark scheme to help you understand what you have done well so far, and how you can do even better in your final assignment)

Here are three things that my Uni Pathways Teacher thought I did well in my draft assignment
•   •   •

Here are three things that my Uni Pathways Teacher thinks that I could do to get a higher mark in my final assignment
•   •   •

Tasks from my Uni Pathways Teacher to do during the feedback tutorial to help me improve
•   •   •
My response:

Actions I will take to improve my final assignment after this tutorial…
•   •   •

Hand in date for my final assignment:	

Tutorial 7 – Final tutorial

What is the Purpose of Tutorial 7?

- To receive feedback and a grade on your final assignment.
- To reflect on the programme including what you enjoyed and what was challenging.
- To ask any questions you may have about university.

Final assignment feedback from my Uni Pathways Teacher

Final mark:	University style grade:
Feedback: Here are three things that my Uni Pathways teacher thought I did well in my final assignment • • •	

Here are three things that my Uni Pathways teacher thinks I should remember for when I am doing this kind of study in the future
• • •

University

What questions do you still have about University after taking part in Uni Pathways?
• • •

Reflecting on Uni Pathways

What did you most enjoy about Uni Pathways?
• • •

What did you find challenging about the programme?	How did you overcome these challenges?
• • •	• • •

Appendix 1 – Referencing correctly

When you get to university, you will need to include references in the assignments that you write, so we would like you to start getting into the habit of referencing in your Brilliant Club assignment. This is really important, because it will help you to avoid plagiarism. Plagiarism is when you take someone else's work or ideas and pass them off as your own. Whether plagiarism is deliberate or accidental, the consequences can be severe. In order to avoid losing marks in your final assignment, or even failing, you must be careful to reference your sources correctly.

What is a reference?

A reference is just a note in your assignment which says if you have referred to or been influenced by another source such as book, website or article. For example, if you use the internet to research a particular subject, and you want to include a specific piece of information from this website, you will need to reference it.

Why should I reference?

Referencing is important in your work for the following reasons:

- It gives credit to the authors of any sources you have referred to or been influenced by.
- It supports the arguments you make in your assignments.
- It demonstrates the variety of sources you have used.
- It helps to prevent you losing marks, or failing, due to plagiarism.

When should you use a reference?

You should use a reference when you:

- Quote directly from another source.
- Summarise or rephrase another piece of work.
- Include a specific statistic or fact from a source.

How do I reference?

There are a number of different ways of referencing, and these often vary depending on what subject you are studying. The most important to thing is to be consistent. This means that you need to stick to the same system throughout your whole assignment. Here is a basic system of referencing that you can use, which consists of the following two parts:

1. **A marker in your assignment**: After you have used a reference in your assignment (you have read something and included it in your work as a quote, or re-written it your own words) you should mark this is in your text with a number, e.g. [1]. The next time you use a reference you should use the next number, e.g. [2].
2. **Bibliography**: This is just a list of the references you have used in your assignment. In the bibliography, you list your references by the numbers you have used, and include as much information as you have about the reference. The list below gives what should be included for different sources.

    a. **Websites** – Author (if possible), title of the web page, website address, [date you accessed it, in square brackets].

    E.g. Dan Snow, 'How did so many soldiers survive the trenches?', http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [11 July 2014].

    b. **Books** – Author, date published, title of book (in italics), pages where the information came from.

    E.g. S. Dubner and S. Levitt, (2006) *Freakonomics*, 7-9.

    c. **Articles** – Author, 'title of the article' (with quotation marks), *where the article comes from* (newspaper, journal etc.), date of the article.

    E.g. Maev Kennedy, 'The lights to go out across the UK to mark First World War's centenary', Guardian, 10 July 2014.

Notes

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