Time as a resource: How much of it should you use to clean your room?

Teacher name:

Pupil name:

Course map

Course Rationale **2**

Timetable and Assignment Submission **3**

Timetable – Tutorials 3

Timetable – Homework Assignments 3

Mark Scheme 3

KS4 Programme 2016-17 – Pupil Feedback Report **4**

Keyword Glossary **5**

Tutorial 1: Understanding the House-cleaning-cookie-baking problem **6**

Homework 1: Baseline Test **10**

Tutorial 2: The graphical method **16**

Homework 2 **21**

Tutorial 3: Understanding higher dimensions **24**

Homework 3 **31**

Tutorial 4: The Standard Form **33**

Homework 4 **35**

Tutorial 5: The Simplex Algorithm **36**

Final Assignment **43**

Tutorial 6: Feedback **46**

Course Rationale

Optimization is a key topic in mathematics as it ubiquitous in everyday life to use available resources to their best potential. As one example we will study the scenario where you and your younger sibling have to use the remaining 3h before your parents return home to clean up the house and to bake cookies after you had friends over. Both cleaning the house and cookies will make your parents happy, but to varying degrees. Given the different speeds at which you and your sibling clean and bake, how much of the house do you clean and how many cookies will you bake for your parents to be the happiest (and least angry)?

Other optimisation problems are the best possible use of various materials in any kind of construction, ranging from buildings or bridges to vehicles and more; or the pricing of product ranges for increased revenue; or the optimal deployment of a given number of workers to achieve and possibly over-attain certain targets. During the course we will cover the topic of Linear Programming in which the cost to be minimised or revenue to be maximised depends linearly on the resources (e.g. time or number of workers).

We will learn to identify Linear Programming problems and how to write them in a mathematical formulation. We will discuss feasibility and optimality of possible solutions and infer mathematical properties of an optimal solution. Equipped with the knowledge of those properties of possible solutions we will be able to solve simple linear programming problems graphically. For more complicated problems a graphical solution will no longer be possible and the necessity of algebraically finding a solution arises. This can be done by using the so-called Simplex Method, which we will learn to use. As part of this we will develop the matrix formulation of a linear programming problem which is designed to keep unnecessary notation to a minimum. At the end of the course we will be able to understand a real world problem such as the House-cleaning-cookie-baking-happiness problem as a linear programming problem, solve it with the Simplex Method and interpret the results of the solution and their implications on a peaceful family life.

Furthermore, the skills you acquire in this course may help you to plan using your time more effectively in other areas as well, for example: How much time of your weekly 15-hour schedule should you spent on each assignment to maximise your final overall grade (some subjects you will find easier and need less time to improve your marks, but they are likely to be high anyway, whereas in areas in which you struggle you need more time, but you could potentially increase your mark by two or more grades).

Linear Programming is a first step into the highly complicated world of optimization problems. In my doctoral thesis I research non-linear problems in infinite dimensions where not even the existence of a solution is certain. In fact, it is known that sometimes no solution exists and this mathematical result has been used to successfully understand physical phenomena in materials such as pattern formation.

Timetable and Assignment Submission

Fill this in with your teacher.

Timetable – Tutorials

Tutorial	Date	Time	Location
1			
2			
3			
4			
5			
6			

Timetable – Homework Assignments

Homework	Description	Due Date
1	The reference problem	
2	The graphical method	
3	Simultaneous equations	
4	The Standard Form	
5	The Simplex Algorithm	

Mark Scheme

Key Skill	1st	2:2
Language	☐ Uses appropriate mathematical vocabulary correctly and effectively.	☐ Generally uses appropriate definitions with incorrect use at times.
Analysis	☐ Correct methods and calculations are executed for derivations and calculations. ☐ Calculations are clearly structured, clear to follow and correct. ☐ Mathematical results are successfully interpreted in the original context.	☐ Has used the correct methods and calculations, but possibly not effectively or with errors. ☐ Calculations are not always easily followed and contain mistakes in places. ☐ Attempts to infer interpretation in original context.
Application	☐ Is able to extend the understanding of the analysis to gain not previously studied insights. ☐ Is able to independently research the effects of alterations of the problem and draw the correct conclusions.	☐ Attempts extension of understanding in unfamiliar territory. ☐ When directed is able to conclude further connections to not previously studied material.

KS4 Programme 2016-17 – Pupil Feedback Report

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at A-Level
2:1	60-69	Performing to a good standard at A-Level
2:2	50-59	Performing to an excellent standard at current key stage
3rd	40-49	Performing to a good standard at current key stage
Working towards a pass	0-39	Performing below a good standard at current key stage
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Name of RIS Teacher			
Title of Assignment	The House-cleaning-cookie-baking-happiness-problem: the case of three siblings		
Name of Pupil			
Name of School			
ORIGINAL MARK/100		**FINAL MARK/100**	
DEDUCTED MARKS		**FINAL GRADE**	

If marks have been deducted (e.g. late submission, plagiarism) the teacher should give an explanation in this section:

Learning Feedback Comment 1 - Language	
What you did in relation to this Key Learning Priority	How you could improve in the future
Learning Feedback Comment 2 - Analysis	
What you did in relation to this Key Learning Priority	How you could improve in the future
Learning Feedback Comment 3 - Application	
What you did in relation to this Key Learning Priority	How you could improve in the future
Resilience Comment	
How you showed learning resilience during the course	How you could build learning resilience in the future

Keyword Glossary

Linear programming	A collection methods to find the optimal solution of problems formulated as a mathematical model whose requirements are represented by linear relationships.
Objective function	The function to be optimised, e.g. the cost of production or the happiness of the returning parents.
Decision variable	The variables of the mathematical model that satisfy linear constraints.
Feasible point	A point that satisfies all given inequalities and equalities.
Feasible set	The collection of points that satisfy all given inequalities and equalities.
Infeasible point	A point that invalidad at least one given inequality or equality.
Optimal solution	The point for which the objective function is optimised.
Extreme point	Vaguely speaking a corner, e.g. verices of convex polygons.
Slack variable	A positive variable added to turn a $\leq \beta$ inequality with $\beta > 0$ into an equality.
Surplus variable	A positive variable substracted to turn a $\geq \beta$ inequality with $\beta > 0$ into an equality.
Linear system of equations	A number of linear equations that need to be solved simultaneously.
Polytope	A geometric object with flat sides.
Convex set	A set for which any two points of the set can be connected by a straight line that is wholly contained in the set.
Hyperplane	An $(n-1)$-dimensional plane through n-dimensional space.
Halfspace	A region of a space on one side of a hyperplane.
Two-dimensional	A problem/set with only two variables.
Higher-dimensional	A problem/set with three or more variables.
Basic feasible solution	A solution that can be read of from a Simplex table by setting a number of variables to zero.
Simplex Algorithm	A method used to solve Linear programming problems.
Entering variable	The variable that will become basic in the Simplex Algorithm.
Departing variable	The variable that leaves the basis to become non-basic, i.e. zero.

Tutorial 1

Overview

What is the learning objective?

- Can you see that the House-cleaning-cookie-baking problem probably has a nontrivial solution?

What are the key questions?

- What is the happiness score for different cleaning/baking strategies?
- Is any of these strategies the best possible one?

What is the homework?

- Further investigate the problem, especially the constraints, and begin to develop an understanding of how to find the solution.

The House-cleaning-cookie-baking problem

Emma and her little brother Tom have been home alone and the house is a little bit messy. In three hours their parents will return and they want to avoid as much trouble as possible. They know that both cleaning the house and freshly baked cookies will make their parents happy. If measured on a scale then each clean square metre of the house would add a happiness level of 5 or every cookie adds a happiness level of 4. As the older sister Emma is faster in both cleaning and baking. She needs 10 minutes for 4 square meters and bakes 21 cookies in an hour, whereas Tom requires 10 minutes for 2 square meters and bakes only 9 cookies in an hour. Tom can only bake cookies under Emma's supervision and he should not spend more than twice the amount of time cleaning of Emma because he will get stroppy otherwise. They have to clean no more than $75\,\mathrm{m}^2$.

How much time should Emma and Tom respectively devote to cleaning the house?

In order to develop an intuition for this problem let us investigate different strategies:

Task 1

1) Emma does all the cleaning while Tom bakes. Once she is finished she helps Tom.
   a) How long does Emma need to clean?
   b) How many cookies can she bake in the remaining time?

c) How much have they cleaned and baked in total and what is the overall happiness score?

2) Emma and Tom both clean until the whole house is spotless, then bake.

a) How long do they need to clean?

b) How many cookies will they bake in the remaining time?

c) What is the overall happiness score?

Now these are just two possible strategies. How would we know if either of them is the best possible one?

Task 2

Find another strategy. Is it better or worse than the two previous ones?

It is tedious to evaluate each strategy from scratch. Using functions we can find a simple formula to calculate

the happiness score. Assume for that we use the following notation:

t_E	... is the time in minutes Emma spends on cleaning	(1)
t_T	... is the time in minutes Tom spends on cleaning	(2)
$c_E = 0.4\,\mathrm{m}^2/\mathrm{min}$	... is the rate at which Emma cleans (0.4 square meters per minute)	(3)
$c_T = 0.2\,\mathrm{m}^2/\mathrm{min}$	... is the rate at which Tom cleans	(4)
$b_E = 0.35\,\mathrm{cookies}/\mathrm{min}$	... is the rate at which Emma bakes (0.35 cookies per minute)	(5)
$b_T = 0.15\,\mathrm{cookies}/\mathrm{min}$	... is the rate at which Tom bakes	(6)
$h_c = 5/\mathrm{m}^2$	... the happiness contribution for each clean square meter	(7)
$h_b = 4\,/\mathrm{cookie}$	... the happiness contribution for each baked cookie	(8)

Find the happiness For example, in time t_E Emma can clean $c_E \times t_E$ square metres. This translates to a happiness contributition of $h_c \times c_E \times t_E$. She then has $180 - t_E$ minutes left to bake cookies.

Task 3

1) Find the happiness contribution of Emma from baking cookies when she spends t_E minutes cleaning.

2) Find the happiness contribution of Tom (both cleaning and baking) when he spends t_T minutes cleaning.

3) Find the overall happiness score $H(t_E, t_T)$ for Emma and Tom spending t_E and t_T minutes cleaning respectively.

$$H(t_E, t_T) = \tag{9}$$

Task 4

1) Using the formula you derived in (9) fill in the following table for the overall happiness factor $H(t_E, t_T)$:

t_T \ t_E	0	30	60	90	120	150	180
0							
30							
60							
90							
120							
150							
180							

If you recognise a pattern, use it, but check your answers.

2) Where is the happiness factor the highest? Is this strategy allowed, e.g. can it always be assured that Tom bakes under Emma's supervision? If not, which requirements are violated?

You should now see that there is a trend in the happiness function, i.e. the happiness is highest when Tom only cleans and Emma only bakes. However, this is not a possible answer due to the given requirements (which one?). To find out which strategies are allowed and which are not is the topic of your homework.

Homework 1

About the Baseline Test

The homework assignment for the first tutorial is a baseline test to see your initial level of attainment in this subject area. The assignment will test for some or all of the subject specific skills that are required later in the final assignment. However, it is shorter than the final assignment and is will be an introduction to the subject as well as a challenge!
Do not worry too much about doing 'well' or 'badly' on the baseline test, it takes into account the fact that you may not be familiar with the subject area. It is designed to help you and your PhD tutor identify where you are at the start of the programme and to help you measure your progress along the way.

The restrictions that are set in the House-cleaning-cookie-baking problem play a crucial part in the solution of the problem. Therefore we have to understand their structure and implications, especially with respect to the variables t_E and t_T for the time Emma and Tom spend cleaning respectively. To summarise, here are the restrictions again:

- Tom can only bake under Emma's supervision.
- Tom should spend no more than twice the amount of time on cleaning than Emma and no more than an hour longer on cleaning than her.
- They have to clean at less than $75\,\mathrm{m}^2$.

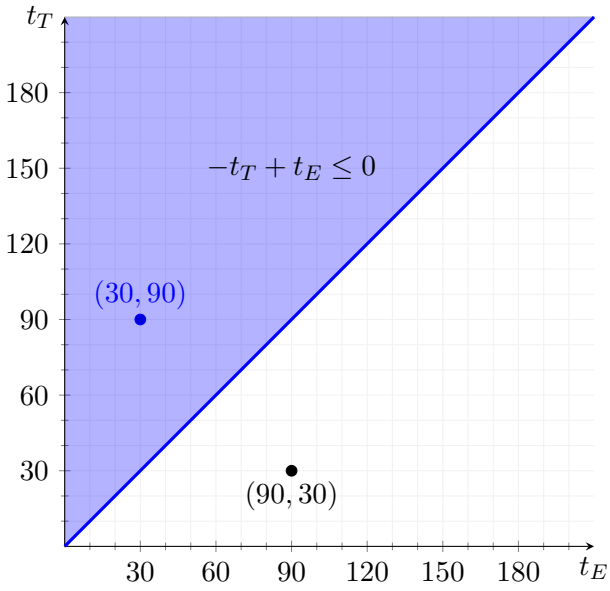
We want to formulate these restriction in terms of the variables t_E and t_T. For example, since Tom can only bake under Emma's supervision, he must spend no more time on baking than Emma (otherwise she is not able to check on him). The times Tom and Emma spend on baking are $180 - t_T$ and $180 - t_E$ respectively. Therefore, we must have that

$$180 - t_T \leq 180 - t_E, \tag{10}$$

which is equivalent to

$$t_T \geq t_E. \tag{11}$$

The graph of this inequality in the t_E, t_T-plane looks like this:

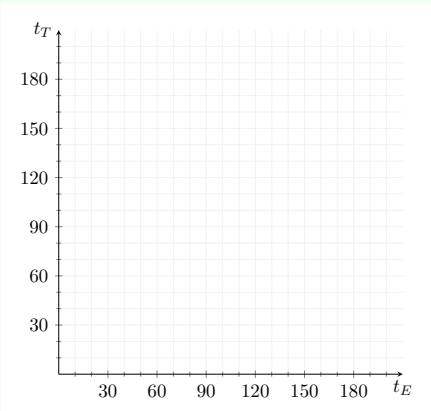


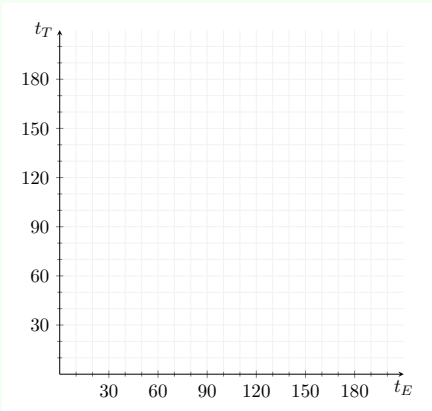
To see this, consider the points for which equality holds, i.e. $t_T = t_E$, which corresponds to the blue diagonal line. The other points for which the inequality holds all lie **on one side** of this line. To determine which side you can simply check a point on either side, for example the point $(t_E, t_T) = (30, 90)$ satisfies the inequality since $90 \geq 30$. Hence, all of the points on the same side of the blue diagonal as $(30, 90)$ satisfy the inequality. Contrastingly, the point $(90, 30)$ does not satisfy the inequality, since $30 < 90$ and so all points below the diagonal do not satisfy the inequality.

Homework 1

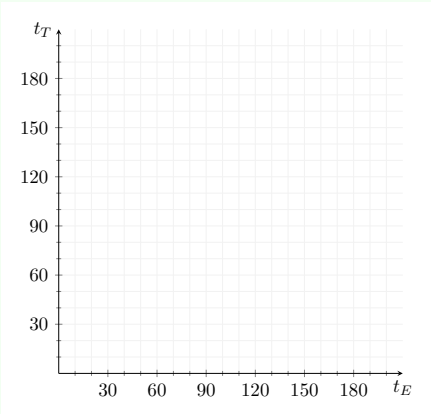
1) Write the requirement 'Tom should spend no more than twice the amount of time on cleaning than Emma and no more than an hour longer on cleaning than her' into two inequality in terms of t_E and t_T.

   Plot all points that satisfy each of the inequalities into the graphs below:





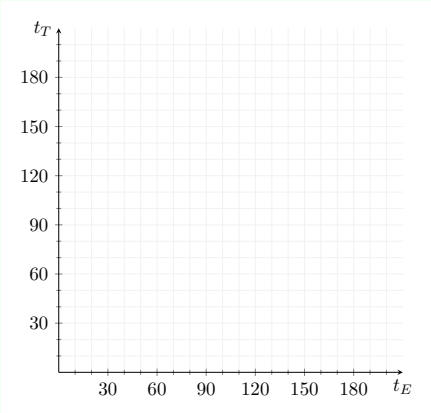
2) Do the same for 'They have to clean at less than 75 m^2'.

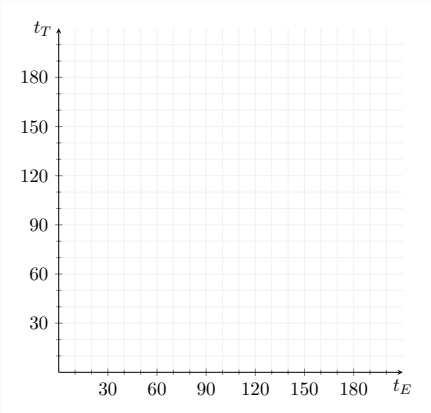


Homework 2

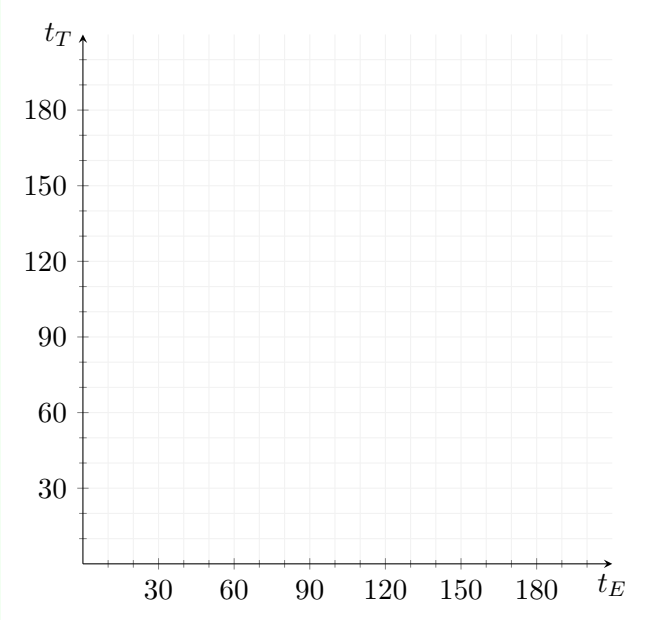
1) Although not explicit at the end of the House-cleaning-cookie-baking problem there are two further restrictions on t_E and t_T. They derive from the fact that their parents return in three hours. What are the restrictions on t_E and t_T?

Draw each in the diagram below.





2) Plot all the restrictions in the graph below. Which set of points satisfies all equations simultaneously?



The corners of the set of points that satisfy all inequalities play an important part. It is therefore relevant to calculate them (and not simply read them off the graph).

Homework 3

Analytically derive the corners of this set. Hint: The corners each lie on a pair of lines defined by the equality part of each inequality. These should give you three equalities in terms of t_E and t_T. Solve each two of them simultaneously.

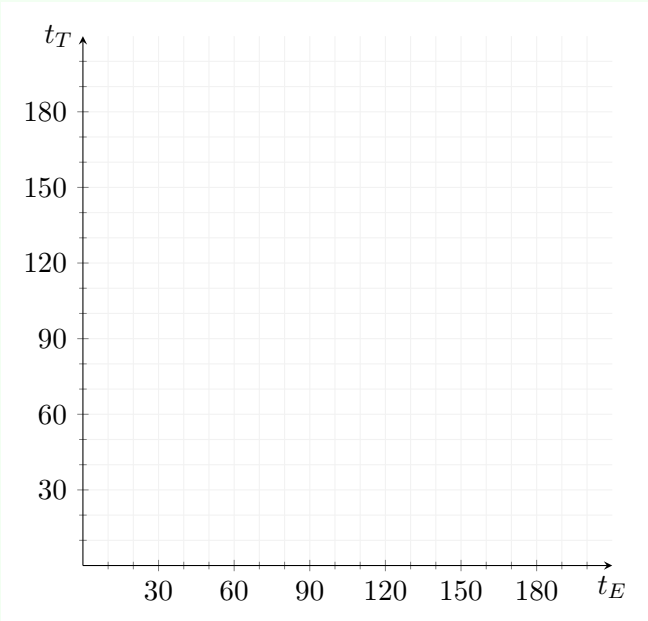
Homework 4

1) Given the happiness function H in (9), what is the happiness score h_i, $i = 1, 2, 3$ at each of the corner points?

2) For each of the happiness values h_i, $i = 1, 2, 3$, of the corners find all the points (t_E, t_T) (whether or not they satisfy the inequalities) for which

$$H(t_E, t_T) = h_i. \tag{12}$$

Hint: Solve the equation you obtain from (12) for t_T and plot.



What can you infer for the solution of the problem?

Tutorial 2

Overview

What are the key questions?

- What are the notions of optimality, feasibility and extremality?

What is the learning objective?

- Formalising the above concepts in mathematical language. Can you talk about linear programming problems in precise mathematical language?

What is the homework?

- Solve the House-cleaning-cookie-baking problem and the rose-growing problem graphically.

In your last homework you may already be able to see that a 'solution' needs to satisfy certain properties. For example, it needs to satisfy all inequalities, which in particular excludes cases $(t_E, t_T) = (30, 90)$ in which Tom cleans for longer than twice the time Emma does. However, not all points (t_E, t_T) that satisfy **all** inequalities qualify as a solution. A solution needs to be 'optimal'. The precise mathematical formulation of this and other notions are the subject of this tutorial.

To begin with, the function that we seek to optimise is called the **objective function** (e.g. the happiness function H), which is a function of the **decision variables** $x_1, \ldots, x_n$ (e.g. $x_1 = t_E$ and $x_2 = t_T$). The equalities or inequalities that need to be satisfied by the decision variables are called **constraints**. A vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ which satisfies **all** constraints is called a **feasible point**.

Task 6

1) In your own words, write down a definition of the notion of the **feasible region** or **solution space**.

2) What is an **infeasible point**?

3) Write down a definition of the **optimal solution** or **optimal point**.

4) What is the **optimum** or **optimal value**?

Task 7

In the House-cleaning-cookie-baking problem, identify...

1) the decision variables...

2) the objective function and state whether it is to be maximised or minimised...

3) the constraints...

The Containers Problem

To produce two types of container, A and B, a company uses two machines, M_1 and M_2. Producing one of container A uses M_1 for 2 minutes and M_2 for 4 minutes. Producing one of container B uses M_1 for 8 minutes and M_2 for 4 minutes. The profit made on each container is £30 for type A and £45 for type B. In the end, we want to determine the production plan that maximises the total profit per hour.

Task 8

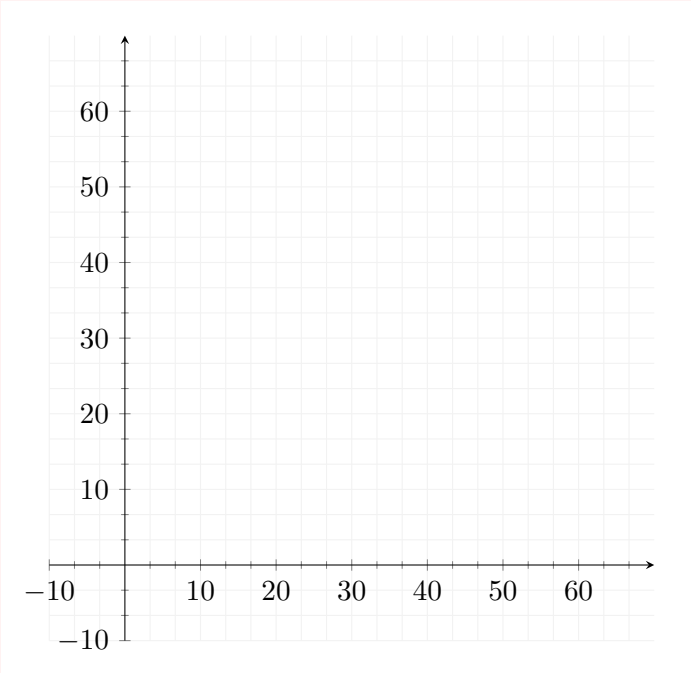
In the Containers Problem, identify...

1) the decision variables...

2) the objective function p and state whether it is to be maximised or minimised...

3) the constraints...

4) Draw the feasible region for the problem. (Label the axes.)

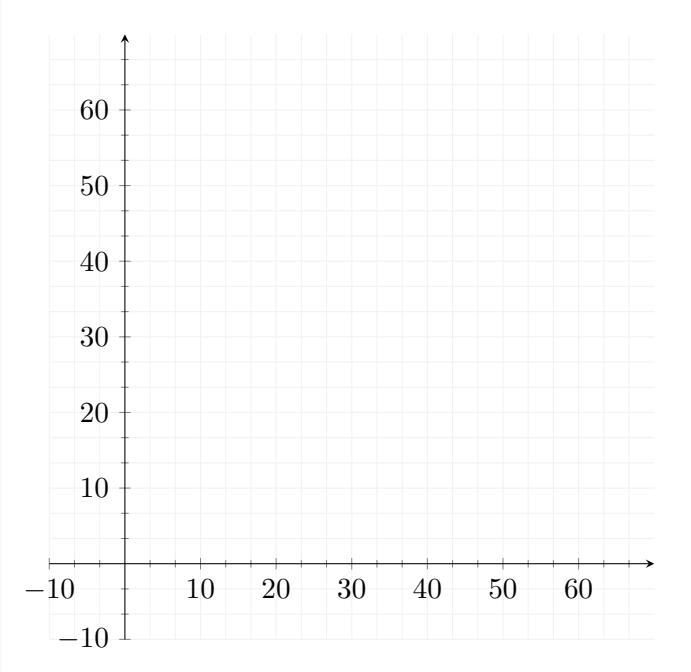


In the homework you have already found all points of a function for which the function value equals a given value (recall the equation $H(t_E, t_T) = h_i$ for $i = 1, 2, 3$). The sets of those points is called a **level set**. For a given real-valued function f and value $c \in \mathbb{R}$ we denote the level set $L_c(f)$ as all those points at which the function f equals c.

Task 9

1) For the Containers Problem, find the level sets $L_{500}(p)$ and $L_{300}(p)$.

2) Plot both level sets in the graph below. What do they have in common?



3) Find the level set $L_c(p)$ for general c. How does it change when c changes?

4) Add the feasible region to the graph above. Explain by reasoning with level sets why the optimal

solution occurs at the point $(A, B) = (10, 5)$.

Note that the point $(A, B) = (10, 5)$ is a corner of the feasible region. In mathematical terms a corner is an **extreme point** of the feasible region. An extreme point of a set is a point which does not lie on the inside of the segment of any two points of the set.

Homework 2

The Rose Growing Problem

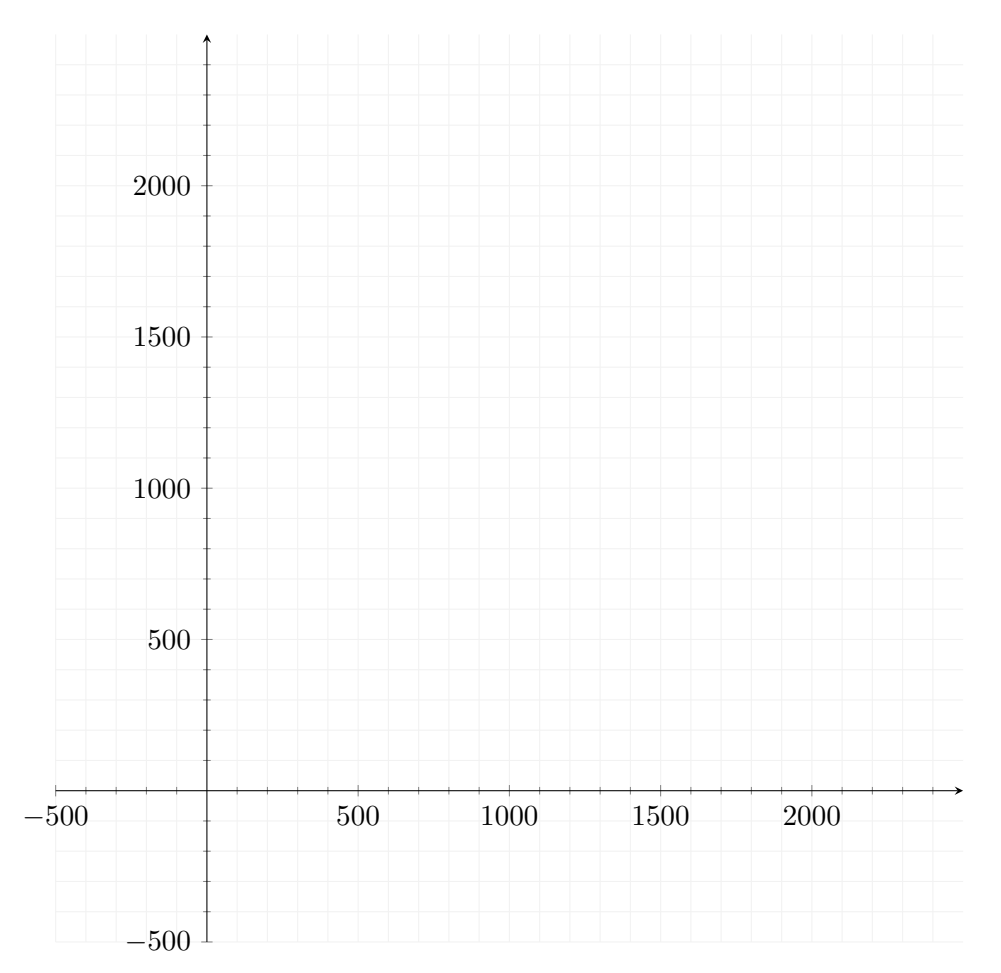
A market gardener grows red and white rose bushes. The red and white bushes require an arear of 5 dm^2 and 4 dm^2 per bush respectively. Each red bush costs £8 per year to grow and each white bush costs £2 per year to grow. The labour needed per year for a red bush is 1 person-hour, whereas for a white bush it is 5 person-hours. The reds each yield a profit of £2 per bush per year and the whites each yield £3 per bush per year. The total land available is at most 6100 dm^2 and the total available finance is £8000. The labour available is at most 5000 person-hours per year.

How many bushes of each type should be planted to maximise the profit?

Homework 5

1) Identify the decision variables, constraints and objective function (and whether it is to be maximised or minimised).

2) Plot the feasible set in the graph below and add three different level sets of the objective function.



3) Find the optimal solution and optimal value of the problem.

So far we have considered problems with **two** decision variables. Because of the two-dimensionality of the problem we were able to plot the problem easily and infer a solution. Have a look at the final assignment.

Homework 6

Looking at the final assignment do the following:

1) Choose a decision variable for the time that Laura helps cleaning. Write down the additional constraints.

2) What is the new happiness (objective) function?

3) Explain in your own words why it is difficult to find a solution by graphical means for this problem.

Tutorial 3

Overview

What is the learning objective?

- Do you understand the limitations of the graphical method in higher dimensions and see the need for algebraic methods?
- Do you understand equations or inequalities in higher dimensions?

What are the key questions?

- What is an n-dimensional space?
- What is a hyperplane in an n-dimensional space?
- What is a convex polytope and how can you find its corners?

What is the homework?

- Compute the intersection of four hyperplanes in a four-dimensional space.

As soon as more than two decision variables are involved a graphical solution approach is not feasible. Instead we have to find other means of finding a solution. This will naturally be an algebraic approach that involves more than two variables (and possibly three) variables. Unfortunately, our human capacity does not allow us to imagine more than three dimensional spaces. However, with our understanding of three dimension we are still able to get an intuition of higher dimensional spaces, even if we cannot imagine them! This tutorial will hopefully help you to get some of that intuition.

Let us first review what the essential ideas are in the 2D case.

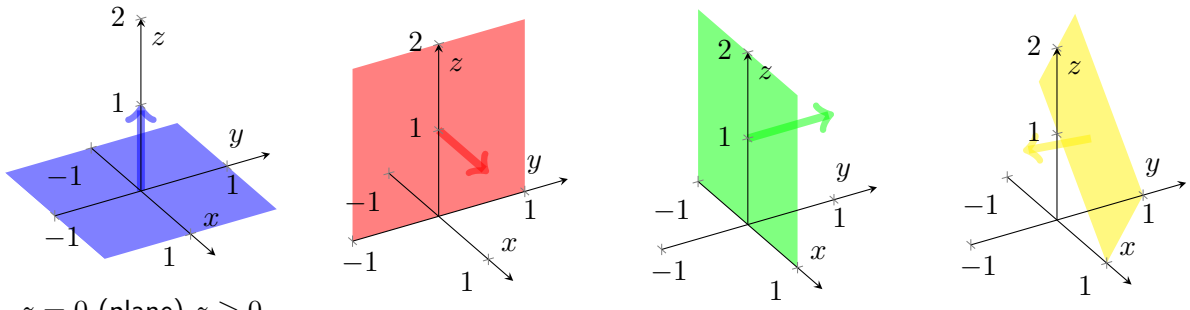
- The key to solving the optimisation problem lies in the feasible set.
- The feasible set is given by the set of all points that satisfy **all** inequalities.
- **One** inequality defines a halfspace.
- The feasible set is the intersection of **all** halfspaces defined by all inequalities.
- A corner of the feasible set can be computed by solving the **two equations** that define it simultaneously.

Fortunately, these ideas all carry over to the 3-dimensional setting. In a 3-dimensional problem we work with three variables, often x, y and z.

- The feasible set is given by all points that satisfy all inequalities, e.g.

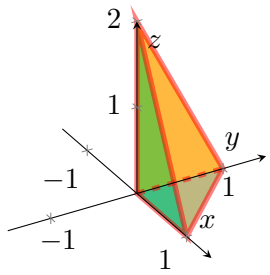
$$\begin{aligned}\mathcal{F} = \{(x, y, z) \ &: 2x + 2y + z \leq 2 \\ &x \geq 0 \\ &y \geq 0 \\ &z \geq 0\}.\end{aligned}$$

- One inequality also defines a halfspace (since it splits the space in two halves).



$z = 0$ (plane) $z \geq 0$ (indicated by arrow) $\quad$ $x \geq 0$ $\quad$ $y \geq 0$ $\quad$ $2x + 2y + z \leq 2$

- The feasible set is the intersection of all halfspaces defined by all inequalities.



$x \geq 0, y \geq 0, z \geq 0, 2x + 2y + z \leq 2$

- A corner of the feasible set can be computed by solving the **three equations** that define it simultaneously.
  - Blue, red and green planes: The 'easiest' corner is defined by the equations $z = 0$ (blue plane), $x = 0$ (red plane) and $y = 0$ (green plane). The point which solves these three equations simultaneously is $(x, y, z) = (0, 0, 0)$.
  - Blue, red and yellow planes: Here we have the three equations:

$$z = 0, \qquad x = 0, \qquad 2x + 2y + z = 2 \tag{13}$$

The first and second equation imply from the third that $y = 1$. Therefore, the point which solves the equations in (13) is $(x, y, z) = (0, 1, 0)$.

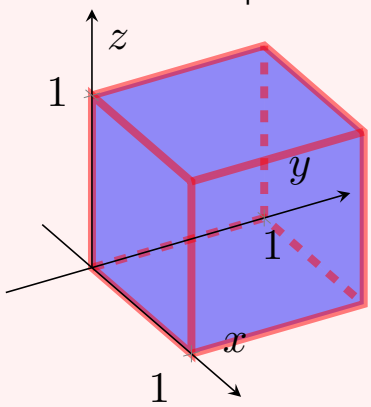
Task 10

1) Verify that the corner defined by the blue, green and yellow planes is $(x, y, z) = (1, 0, 0)$.

2) Verify that the corner defined by the red, green and yellow planes is $(x, y, z) = (0, 0, 2)$.

Task 11

Write down the inequalities that define the unit cube. (Hint: The cube has six sides.)

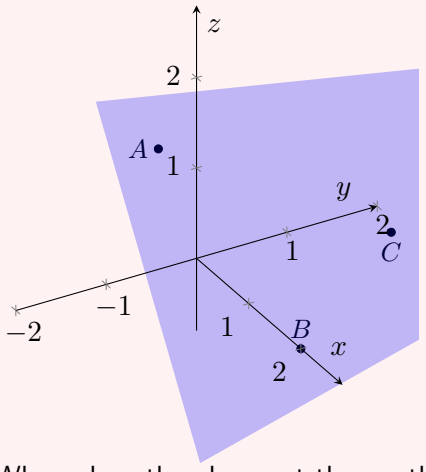


Unfortunately, all humans seem unable to visualise or imagine more than three dimensions. However, what we have seen in two and three dimensions is generalisable. A hyperplane is a subspace of **one dimension less** than its ambient space. A hyperplane is always given by **one** equation in the variables of the ambient space. If a space is 3-dimensional then its hyperplanes are the 2-dimensional planes, while if the space is 2-dimensional, its hyperplanes are the 1-dimensional lines.

Task 12

Find the coefficients a, b and c so that the equation $ax + by + cz = 6$ is given by the indicated plane. The three points A, B and C lie in the plane.

$$A = (1, -1, 2)$$
$$B = (2, \quad 0, 0)$$
$$C = (2, \quad 1, 1)$$



Where does the plane cut the x-, the y- and the z-axis respectively?

The above example was quite easy to solve as you could see that $a = 3$ by using point B. You were then left with two equations in two the two variables b and c and you should know how to solve this. We are now going to

develop a system that lets you solve general systems of three variables. Assume we wanted to find the solution of the three equations

$$x + \frac{2}{3}y - \frac{1}{3}z = 3 \qquad -2x + y + z = -4 \qquad x - y + 2z = -1.$$

Task 13

Below is a method that finds the solution. Make notes to the side of this diagram as to what is going on!

$$\begin{array}{rcrcrcr} x & + & \frac{2}{3}y & - & \frac{1}{3}z & = & 3 \\ -2x & + & y & + & z & = & -4 \\ x & - & y & + & 2z & = & -1 \end{array} \quad \begin{array}{l} \times 2 \\ \times(-1) \end{array} \longrightarrow \quad \begin{array}{rcrcrcr} x & + & \frac{2}{3}y & - & \frac{1}{3}z & = & 3 \\ & + & \frac{7}{3}y & + & \frac{1}{3}z & = & 2 \\ & - & \frac{5}{3}y & + & \frac{7}{3}z & = & -4 \end{array} \quad \times\tfrac{3}{7}$$

$$\longleftarrow$$

$$\begin{array}{rcrcrcr} x & + & \frac{2}{3}y & - & \frac{1}{3}z & = & 3 \\ & & y & + & \frac{1}{7}z & = & \frac{6}{7} \\ & - & \frac{5}{3}y & + & \frac{7}{3}z & = & -4 \end{array} \quad \times\tfrac{3}{5} \longrightarrow \quad \begin{array}{rcrcrcr} x & + & \frac{2}{3}y & - & \frac{1}{3}z & = & 3 \\ & & y & + & \frac{1}{7}z & = & \frac{6}{7} \\ & & & & \frac{18}{7}z & = & -\frac{18}{7} \end{array} \quad \times\tfrac{7}{18}$$

$$\longleftarrow$$

$$\begin{array}{rcrcrcr} x & + & \frac{2}{3}y & - & \frac{1}{3}z & = & 3 \\ & & y & + & \frac{1}{7}z & = & \frac{6}{7} \\ & & & & z & = & -1 \end{array} \quad \begin{array}{l} \times\frac{1}{3} \\ \times(-\frac{1}{7}) \end{array} \longrightarrow \quad \begin{array}{rcrcrcr} x & + & \frac{2}{3}y & & & = & \frac{8}{3} \\ & & y & & & = & 1 \\ & & & & z & = & -1 \end{array} \quad \times(-\tfrac{2}{3})$$

$$\longleftarrow$$

$$\begin{array}{rcrcrcr} x & & & & & = & 2 \\ & & y & & & = & 1 \\ & & & & z & = & -1 \end{array}$$

What is the solution of the problem?

Finally, we notice that the variables itself are just additional load in notation. By using a table we can considerably reduce the amount of notation and make the problem visually more appealing. Compare each step to ensure yourself that both procedure are exactly the same!

x	y	z	
1	$\frac{2}{3}$	$\frac{1}{3}$	3
-2	1	1	-4
1	1	2	-1

$\times 2$, $\times(-1)$

x	y	z	
1	$\frac{2}{3}$	$\frac{1}{3}$	3
0	$\frac{7}{3}$	$\frac{1}{3}$	2
0	$\frac{5}{3}$	$\frac{7}{3}$	-4

$\times \frac{3}{7}$

x	y	z	
1	$\frac{2}{3}$	$\frac{1}{3}$	3
0	1	$\frac{1}{7}$	$\frac{6}{7}$
0	$\frac{5}{3}$	$\frac{7}{3}$	-4

$\times \frac{3}{5}$

x	y	z	
1	$\frac{2}{3}$	$\frac{1}{3}$	3
0	1	$\frac{1}{7}$	$\frac{6}{7}$
0	0	$\frac{18}{7}$	$-\frac{18}{7}$

$\times \frac{7}{18}$

x	y	z	
1	$\frac{2}{3}$	$-\frac{1}{3}$	3
0	1	$\frac{1}{7}$	$\frac{6}{7}$
0	0	1	-1

$\times \frac{1}{3}$, $\times(-\frac{1}{7})$

x	y	z	
1	$\frac{2}{3}$	0	$\frac{8}{3}$
0	1	0	1
0	0	1	-1

$\times(-\frac{2}{3})$

x	y	z	
1	0	0	2
0	1	0	1
0	0	1	-1

Solving a system of linear equations

Given: A system of linear equations in the variables $x_1, \ldots, x_n$, i.e.

$$\begin{array}{ccccccccc} a_1^1 x_1 & + & a_2^1 x_2 & + & \ldots & + & a_n^1 x_n & = & b_1 \\ a_1^2 x_1 & + & a_2^2 x_2 & + & \ldots & + & a_n^2 x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_1^n x_1 & + & a_2^n x_2 & + & \ldots & + & a_n^n x_n & = & b_n \end{array} \tag{14}$$

Wanted: A solution $x_1, \ldots, x_n$ that satisfies all equations simultaneously.

(i) Write the system in tabular form:

x_1	x_2	$\dots$	x_n	
a_1^1	a_2^1	$\dots$	a_n^1	b_1
a_1^2	a_2^2	$\dots$	a_n^2	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_1^n	a_2^n	$\dots$	a_n^n	b_n

$\times \frac{1}{a_1^1}$

(ii) Ensure $a_1^1 \neq 0$, otherwise swap the first row with any other row j where $a_1^j \neq 0$.

(iii) Divide the first row by a_1^1 to obtain:

x_1	x_2	$\dots$	x_n	
1	$\frac{a_2^1}{a_1^1}$	$\dots$	$\frac{a_n^1}{a_1^1}$	$\frac{b_1}{a_1^1}$
a_1^2	a_2^2	$\dots$	a_n^2	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_1^n	a_2^n	$\dots$	a_n^n	b_n

$\times(-a_1^2)$

$\times(-a_1^n)$

(iv) Use the first row to make all entries below 1 to zero. To make the first entry in the j-th row to zero take a a_1^j multiple of the first row and substract it from the j-th row to obtain:

x_1	x_2	$\dots$	x_n	
1	$\frac{a_2^1}{a_1^1}$	$\dots$	$\frac{a_n^1}{a_1^1}$	$\frac{b_1}{a_1^1}$
0	$a_2^2 - \frac{a_2^1}{a_1^1}a_1^2$	$\dots$	$a_n^2 - \frac{a_n^1}{a_1^1}a_1^2$	$b_2 - \frac{b_1}{a_1^1}a_1^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	$a_2^n - \frac{a_2^1}{a_1^1}a_1^n$	$\dots$	$a_n^n - \frac{a_n^1}{a_1^1}a_1^n$	$b_n - \frac{b_n}{a_1^1}a_1^n$

(v) Disregard first row and column and repeat steps ii to iv until the table looks like this:

x_1	x_2	x_3	$\dots$	x_n	
1	$\tilde{a}_2^1$	$\tilde{a}_3^1$	$\dots$	$\tilde{a}_n^1$	$\tilde{b}_1$
0	1	$\tilde{a}_3^2$	$\dots$	$\tilde{a}_n^2$	$\tilde{b}_2$
$\vdots$	0	1	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\tilde{a}_n^{n-1}$	$\vdots$
0	0	$\dots$	0	1	$\tilde{b}_n$

$\times(-\tilde{a}_n^1)$

$\times(-\tilde{a}_n^2)$

$\times(-\tilde{a}_n^{n-1})$

(vi) Apply the same method to turn all entries **above** the ones into zeros, starting with the last column of variables:

x_1	x_2	x_3	$\dots$	x_n	
1	$\tilde{a}_2^1$	$\tilde{a}_3^1$	$\dots$	0	$\tilde{b}_1 - \tilde{a}_n^1 \tilde{b}_1$
0	1	$\tilde{a}_3^2$	$\dots$	0	$\tilde{b}_2 - \tilde{a}_n^2 \tilde{b}_2$
$\vdots$	0	1	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	0	$\tilde{b}_{n-1} - \tilde{a}_n^{n-1} \tilde{b}_n$
0	0	$\dots$	0	1	$\tilde{b}_n$

(vii) Proceed similarly from columns $n-1$ back to the second column until your table looks like this:

x_1	x_2	x_3	$\dots$	x_n	
1	0	0	$\dots$	0	$\hat{b}_1$
0	1	0	$\dots$	0	$\hat{b}_2$
$\vdots$	0	1	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	0	$\hat{b}_{n-1}$
0	0	$\dots$	0	1	$\tilde{b}_n$

The solution of the system of equations is given in the last column.

Homework 3

Homework 7

Solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 9 \\ x_1 + x_2 - x_3 - 2x_4 &= 1 \\ 2x_1 - x_2 + x_3 \phantom{{}+2x_4} &= 4 \\ -x_1 + x_2 + 2x_3 + 2x_4 &= 0. \end{aligned}$$

Homework 8

Not all systems of linear equations always have a solution. For example, this done does not have a solution:

$$\begin{aligned} 2x + \ \ y - \ \ z &= 3 \\ x + \ \ y + 2z &= 4 \\ 2x + \ \ y - \ \ z &= 5. \end{aligned}$$

Algebraically speaking the reason is if both the first and third equation are true at the same time, then this implies that $3 = 5$, which is clearly false. Therefore, there cannot be a solution that solves all three equations simultaneously.

Explain in **geometric** words why the system does not have a solution. (Hint: You know that each equation represents a hyperplane. Describe how the first and third hyperplane lie to each other.)

Tutorial 4

Overview

What is the learning objective?

- Can you write any linear programming problem in its standard form?

What are the key questions?

- What characterises the 'standard form'?
- What is a slack variable?
- What is a surplus variable?
- What is a basic feasible solution?

What is the homework?

- Write the House-cleaning-Cookie-Baking problem (the case two siblings) and the Rose-growing problem into the standard form.

We know from Tutorial 1 that the key to the optimisation lies in the constraints of the problem that define the feasible set. From Tutorial 2 we know that the optimal solution occurs at a corner (or extreme point) of the feasible set and we exploited this in the graphical method. In higher dimensions we can no longer visualise the feasible set, but we learned how to find intersections of hyperplanes. In principle, if we have three variables and a number of inequalities that define the feasible set, we could simply compute all corners of this set, evaluate the objective function on each corner and find the corner with the lowest value – it will be the optimal solution. However, this method can easily get very cumbersome as the number of inequalities (and therefore the number of corners) grows. Furthermore, it is not simple to see anymore, which intersection of hyperplanes are actually corners of the feasible set. The standard form is an equivalent way of writing the linear programming problem with two implications:

- It will be easy to see which intersections of hyperplanes are corners of the feasible set.
- Starting at one corner, it will become apparent which corners to investigate next, rather than checking all corners. This will save us a lot of work in the end.

Definition: Standard form

A linear programming problem is in **standard form** if we seek to

$$\textbf{maximise } z = c_1x_1 + \ldots + c_nx_n$$

subject to

$$\begin{array}{ccccccc} a_1^1 x_1 & + & \dots & + & a_n^1 x_n & = & b_1 \\ a_1^2 x_1 & + & \dots & + & a_n^2 x_n & = & b_2 \\ \vdots & + & \dots & + & \vdots & = & \dots \\ a_1^m x_1 & + & \dots & + & a_n^m x_n & = & b_m \end{array} \tag{15}$$

and

$$x_i \geq 0, \; i = 1, \dots, n$$

and where b_i are all positive numbers. In other words, we seek to maximise the objective function z among all points $(x_1, \dots, x_n)$ with only **positive** entries and that satisfy the equality constraints (15).

Note that all our problems so far have inequality constraints rather than equality constraints. In the following example and task we will practice changing an inequality into an equality using an additional variable while ensuring that all involved variables obey the positivity constraint.

Task 14

Introduce either slack or surplus variables to write the following inequalities as equalities with non-zero right hand side and **all** variables greater or equal to zero.

(a) $3x_1 - 5x_2 \leq 2,$ such that $x_1 \geq 0, x_2 \geq 0$

(b) $x - y + 2z \geq -1,$ such that $x \geq 0, y \geq 0, z \geq 0$

(c) $5x - 2y \leq 3,$ such that $x \geq 0, y \leq 0$

(d) $x_1 + 3x_2 - x_3 \leq 2,$ such that $x_1 \leq 1, x_2 \geq 0, x_3 \geq 0$

Homework 4

Homework 9

1) Write the House-cleaning-cookie-baking problem into the standard form.

2) Write the Rose-growing problem into the standard form.

Tutorial 5

Overview

What is the learning objective?

- Can you apply the Simplex Algorithm to solve a higher dimensional linear programming problem?

What are the key questions?

- Is the solution optimal?
- How do you identify the entering variable?
- How do you identify the departing variable?

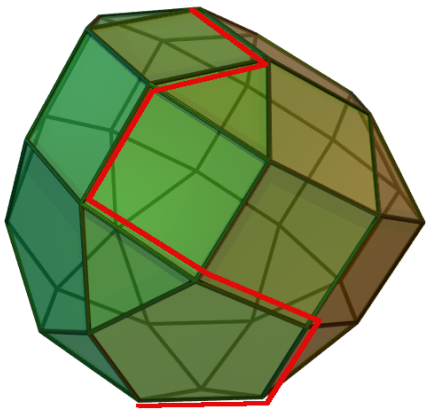
What is the homework?

- Complete the final assignment.

In the last tutorial we saw how to write linear programming problems into the 'standard form'. In the homework you will have started getting an understanding of the extreme points (i.e. corners) of the feasible set. We know that the optimum is attained at one of the extreme points of this set and the idea of the Simplex Algorithm is simple:

- to move from one extreme point to a neighbouring one, ensuring that the objective function always increases until no further increase is possible and the optimal value has been reached.

Pictorally, this can be interpreted with the following picture, where the green polytope represents the feasible region and the red line indicates the journey along the extreme points (vertices) of the polytope:



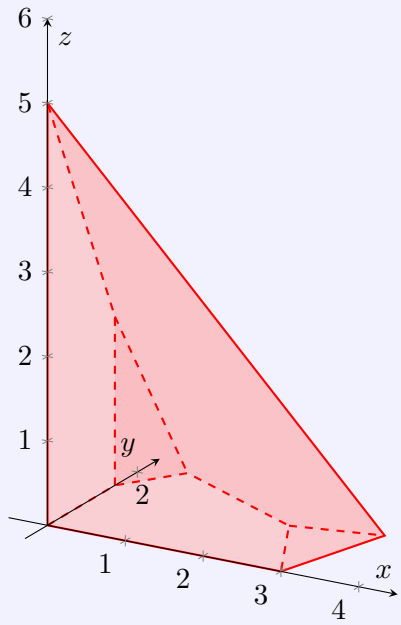
We will explain the particulars of the Simplex Algorithm on the following example:

Example problem

Maximise $z = 2x_1 + 4x_2 - 5x_3$ subject to

$$x_1 + 2x_2 + x_3 \leq 5, \qquad 2x_1 + x_2 - 4x_3 \leq 6, \qquad 3x_1 - 2x_2 \geq -3$$

and $x_i \geq 0$, $i = 1, 2, 3$. The feasible region given by the six inequalities is illustrated below.



In standard form the problem is the following:
Maximise $z = 2x_1 + 4x_2 - 5x_3$ subject to

$$x_1 + 2x_2 + x_3 + s_1 = 5, \qquad 2x_1 + x_2 - 4x_3 + s_2 = 6, \qquad -3x_1 + 2x_2 + s_3 = 3$$

all $x_i \geq 0$, $i = 1, 2, 3$.

The first step that we take is to write all these equations into the table that we know from Tutorial 3. For example, we can write:

x_1	x_2	x_3	s_1	s_2	s_3	
1	2	1	1	0	0	5
2	1	-4	0	1	0	6
-3	2	0	0	0	1	3

Furthermore, we will now also consider z as a variable and consider the function to be maximised as a constraint, which then reads as

$$z - 2x_1 - 4x_2 + 5x_3 = 0$$

and add it to the table, which then becomes:

z	x_1	x_2	x_3	s_1	s_2	s_3	
0	1	2	1	1	0	0	5
0	2	1	-4	0	1	0	6
0	-3	2	0	0	0	1	3
1	-2	-4	5	0	0	0	0

Finally, we add another column 'B' which will help us keep track of what we are doing. It is called the **base column**.

B	z	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	0	1	2	1	1	0	0	5
s_2	0	2	1	-4	0	1	0	6
s_3	0	-3	2	0	0	0	1	3
z	1	-2	-4	5	0	0	0	0

Notice that we have three equations, but six variables. You should know that usually three variables are uniquely defined by three equations, but now we have more than that. The idea of the algorithm is to always choose all but three variables to be set to zero (three comes from the fact that we had three inequalities). For a simple start, we can choose $x_1 = x_2 = x_3 = 0$. We call these variables **non-basic**. The others, s_1, s_2, s_3 are **basic**, which is why they are in the 'B' column.

Task 15

Choose $x_1 = x_2 = x_3 = 0$. Solve for s_1, s_2 and s_3. What do you find? Compare to the 'B' column.

However, choosing x_1, x_2 and x_3 to equal zero was not optimal, because the z row (the last row) has negative entries for x_1 and x_2.

Task 16

Discuss why the negative entries in the z-row imply that setting $x_1 = x_2 = x_3 = 0$ is not optimal.

There are 30 choices for setting three of the six variables to zero, each corresponding to the intersection of three of six hyperplanes, making up the vertices. However, we do not want to guess which of the 30 choices is the right one, but arrive there in a systematic manner. The function z can be increased, if we increase x_1 or x_2. Because the coefficient for x_2 is larger than that for x_1 in z we choose to increase x_2. By that we mean that we free x_2, i.e. remove our previous choice of setting it to zero, and making it non-zero. Therefore, x_2 enters the basis, i.e. becomes basic, and it is therefore called the **entering variable**. To decide which of the three others, s_1, s_2 or s_3 will become non-basic we need to look at both the x_2 column and the last column and compute the quotients of the two in each row:

B	z	x_1	x_2	x_3	s_1	s_2	s_3		θ
s_1	0	1	2	1	1	0	0	5	$5/2$
s_2	0	2	1	-4	0	1	0	6	$6/1$
s_3	0	-3	2	0	0	0	1	3	$3/2$
z	1	-2	-4	5	0	0	0	0	

smallest!

In the same row as the lowest coefficient is s_3, which we make non-basic. So we want the column of x_2 to equal to that of s_3 currently. We do this in two steps. First we divide the fourth row by 2, creating a one in the dark red shaded area.

B	z	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	0	1	2	1	1	0	0	5
s_2	0	2	1	-4	0	1	0	6
s_3	0	$-\frac{3}{2}$	1	0	0	0	$\frac{1}{2}$	$\frac{3}{2}$
z	1	-2	-4	5	0	0	0	0

$\times(-2)$
$\times(-1)$
$\times 4$

Now we want to make all other entries in the x_2 column to zero by adding multiples of the row to the others as indicated by the arrows. The resulting table is this:

B	z	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	0	4	0	1	1	0	-1	2
s_2	0	$\frac{7}{2}$	0	-4	0	1	$-\frac{1}{2}$	$\frac{9}{2}$
x_2	0	$-\frac{3}{2}$	1	0	0	0	$\frac{1}{2}$	$\frac{3}{2}$
z	1	-8	0	5	0	0	2	6

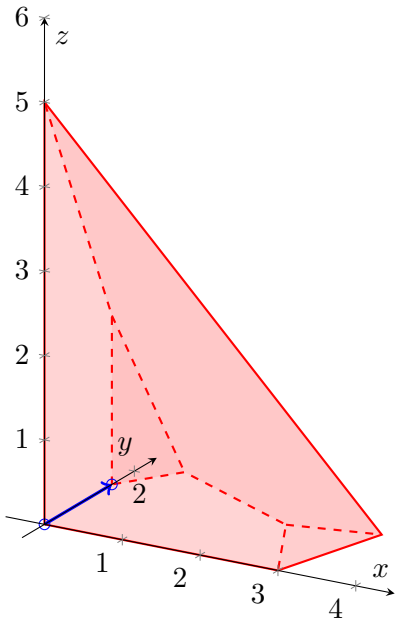
Notice that we have now labelled x_2 in the base column 'B', whereas s_3 is no longer basic. Therefore, s_3 is called the leaving variable.

Task 17

1) Now x_1, x_3 and s_3 are non-basic, i.e. we choose $x_1 = x_3 = s_3 = 0$. What are the values for x_2, s_1 and s_2? Is the point (x_1, x_2, x_3) feasible for those values?

2) If we had chosen any other row to make it into a one and the rest to zero, what would have happened? Try it for the second row, i.e. the one starting with s_2! Would (x_1, x_2, x_3) be feasible?

Setting focussing only one x_1, x_2 and x_3 we find the new vertex $(0, \frac{3}{2}, 0)$. We started from $(0, 0, 0)$, so indeed, the Simplex Algorithm moved on one vertex to a neighbouring one and also increased the value of z by doing so!



This vertex is not optimal as the last row of the last table still includes a negative number. So we want x_1 to enter the basis. Again we compute the quotients of the x_1 column and the last column.

B	z	x_1	x_2	x_3	s_1	s_2	s_3		θ
s_1	0	4	0	1	1	0	-1	2	$\frac{1}{2}$
s_2	0	$\frac{7}{2}$	0	-4	0	1	$-\frac{1}{2}$	$\frac{9}{2}$	$\frac{9}{7}$
x_2	0	$-\frac{3}{2}$	1	0	0	0	$\frac{1}{2}$	$\frac{3}{2}$	
z	1	-8	0	5	0	0	2	6	

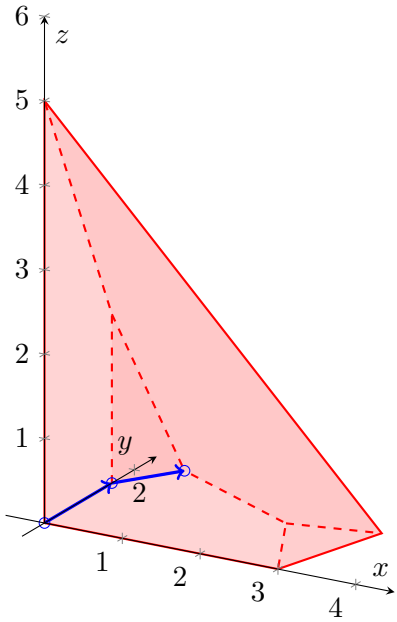
Task 18

1) Complete the final table. Which variables are now basic? Which are non-basic?

2) Notice that we disregarded the negative entry in the x_1 column. Try what would have happened if we

had chosen to use this row instead!

From your final table you should be able to see that we moved from $(0, \frac{3}{2}, 0)$ to $(\frac{1}{2}, \frac{9}{4}, 0)$ as shown below. We have now found the solution as z cannot be increased further!



To summarise the algorithm:

The Simplex algorithm

1. From the standard form write down the initial table including the 'B' and z column and the z row.
2. If the z row contains a negative number, we have not found the optimum yet.
   i. Choose the column that contains the most negative number in the z row.
   ii. Compute the quotient of the last column with the chosen column, disregarding non-positive entries.
   iii. Make the entry of the **smallest positive** quotient in the chosen column to one and everything else to zero.
3. If the z row contains no negative number we have found the optimum, else return to step 2.

Task 19

Solve the House-cleaning-cookie-baking problem for two siblings using the Simplex Algorithm.

Final Assignment

The House-cleaning-cookie-baking problem: The case of three siblings

It is four years later and Emma and Tom now have a younger sister Laura and they moved into a larger house.

Again, the parents return home in three hours and they can make them happy by either cleaning the house or baking cookies. Both Tom and Laura need supervision by Emma while baking. Like last time, Tom gets annoyed if he cleanes more than twice the time as Emma does or an hour more than her. Laura gets upset if she cleans longer than both Tom and Emma together.

They have to clean an area of at most $150\,\mathrm{m}^2$. Emma, Tom and Laura now clean at rates $0.6\,\mathrm{m}^2/\mathrm{min}$, $0.4\,\mathrm{m}^2/\mathrm{min}$ and $0.2\,\mathrm{m}^2/\mathrm{min}$ respectively and they bake 0.45 cookies/min, 0.25 cookies/min and 0.1 cookies/min.

Every clean square meter makes a happiness contribution of 5.5, every baked cookie of 4.

What is the optimal time for the siblings to spend on cleaning and baking?

Assume the happiness contribution for each clean square meter was changed by the amount δ, i.e. $5.5 + \delta$. Find new happiness value H' in terms of the old one H and δ. Then, using this in your last table from the simplex algorithm, change H to H'.

What values can δ take for the solution remain optimal?

(Hint: Use the other rows to create zeros in the H' row for all basic columns. Remember then that a solution is only optimal if **all** entries of the H' row are non-negative.)

Tutorial 6

Overview

What is the learning objective?

- Reflect on skills learned on the programme with a view to encouraging resilience.
- Receive feedback on final assignment.

What are the key questions?

- What strengths have I demonstrated in my work and what areas of development are there left for me to address?
- What steps do I need to take to improve my academic output?

Activity 1: Marking your own

Using the mark scheme, identify three things you did well and three things you could improve.

What went well?

1.
2.
3.

What could be improved?

1.
2.
3.

Activity 2: Responding to feedback

Read through your feedback sheet and highlight on your assignment where you received positive feedback (one colour) and areas for improvement (different colour). You might find it helpful to annotate this with a brief comment or two.

Activity 3: Looking Forwards – Think, Pair, Share

1. Think about how you can use this feedback in your future school work.
2. Talk to your partner about how you both plan to make changes to your school work in the future as result of this feedback.
3. Share your ideas as a group.