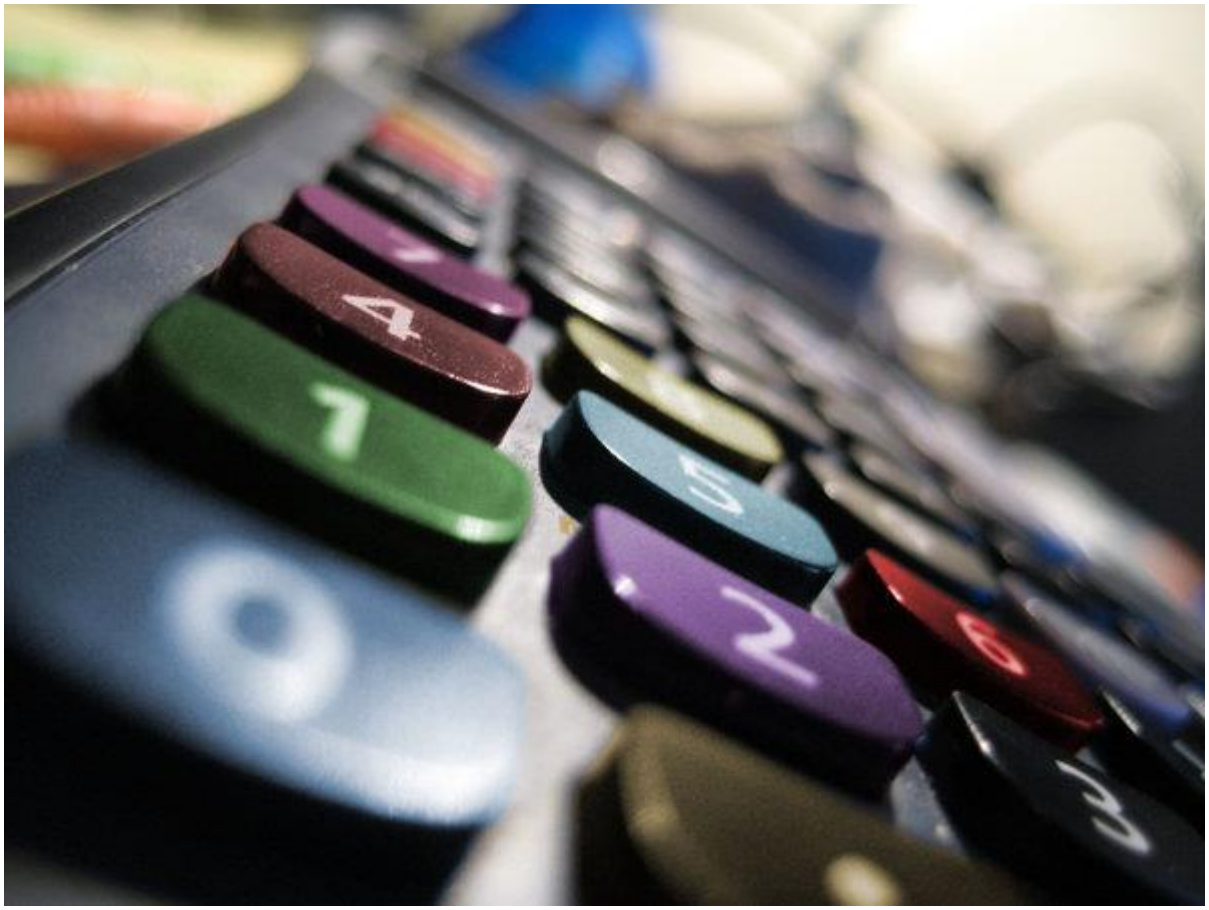
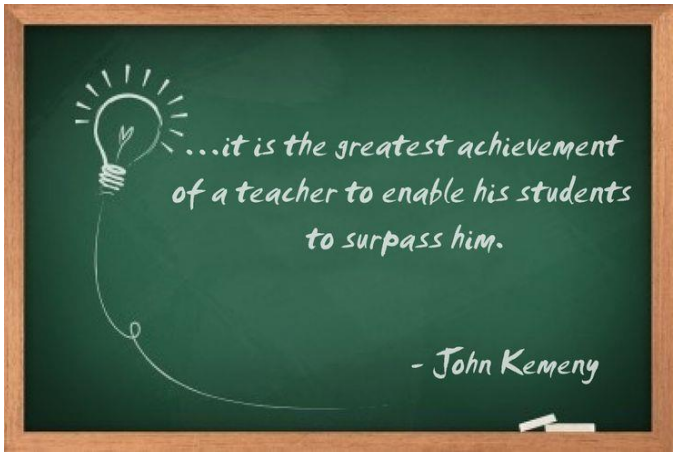


The Scholars Programme

So you think you can add up numbers, don't you?

[Autumn 2015]

Tutor Name



Pupil Name

The Scholars Programme – [Autumn 2016] – Pupil Feedback Report

Grade	Marks	What this means
1st	70+	Performing to an excellent standard at first year undergraduate level
2:1	60-69	Performing to a good standard at first year undergraduate level
2:2	50-59	Performing to an excellent standard at GCSE level
3rd	40-49	Performing to a good standard at GCSE level
Workings towards a pass	0-39	Workings towards a pass
Did not submit	DNS	No assignment received by The Brilliant Club

Lateness	
Any lateness	10 marks deducted
Plagiarism	
Some plagiarism	10 marks deducted
Moderate plagiarism	20 marks deducted
Extreme plagiarism	Automatic fail

Name of PhD Tutor			
Title of Assignment	Mastering basic facts in addition		
Name of Pupil			
Name of School	The Langley Academy		
ORIGINAL MARK / 100		**FINAL MARK / 100**	
DEDUCTED MARKS		**FINAL GRADE**	

If marks have been deducted (e.g. late submission, plagiarism) the PhD tutor should give an explanation in this section:

Learning Feedback Comment 1 - *Key Learning Priority*_____	
What you did in relation to this Key Learning Priority	**How you could improve in the future**
Learning Feedback Comment 2 – *Key Learning Priority*_____	
What you did in relation to this Key Learning Priority	**How you could improve in the future**
Learning Feedback Comment 3 – *Key Learning Priority*_____	
What you did in relation to this Key Learning Priority	**How you could improve in the future**
Resilience Comment	
How you showed learning resilience during the course	**How you could build learning resilience in the future**

Contents

Course Rationale Page 4
Mark Scheme Page 5
Glossary of Keywords Page 6
Tutorial 1 – Sum of finite sequences Page 7
- Activity 1 Page 10
- Series and partial sums Page 12
- Tutorial 1 Homework – Baseline Test Page 16

Tutorial 2 – Particular finite sums and their properties Page 17
- Particular partial sums Page 17
- Activity 2 Page 18
- Activity 3 Page 20
- Tutorial 2 Homework Page 20

Tutorial 3 – Infinite sequences and their series Page 22
- Geometric Series Page 25
- Harmonic Series Page 25
- Activity 4 Page 29
- Tutorial 3 Homework Page 32

Tutorial 4 – Graph of simple functions Page 33
- Activity 5 Page 38
- Ordered pairs Page 39
- Tutorial 4 Homework Page 41

Tutorial 5 – Areas as limit of sums Page 43
- Introduction to Integration Page 43
- Activity 6 Page 45
- Final Assignment Page 47

Tutorial 6 – Feedback and Reflection Time Page 51

University Applications Guidance Page 52

Course Rationale

There is a big difference between memorising and understanding mathematical formulae. Knowing deeply maths facts frees up the mind to solve real math problems. Example: what is the sum of the first 100 numbers?
There is a well-known story about the mathematician Karl Friedrich Gauss when he was in elementary school. His teacher got mad at the pupils and told them to add all the numbers from 1 to 100 and give him the answer by the end of the class. About 30 seconds later Gauss gave him the answer!
In this course we will learn efficient ways to calculate sums of this kind and we will see how important they are for finding the area of two-dimensional regions which are not polygons (regions bounded by parts of a straight line). In particular, we will learn how to calculate the area under a curve as a limit of sums of simpler areas. Moreover, we will look at the strange behaviour of infinite sums (series) of numbers.

The course is a mixture of algebra and analysis and gives a taste of the content covered in the first courses of Analysis at university level.
As soon as we become confident with the notation relative to the theory of sequences of numbers and their series, we will develop the ability to connect different parts of mathematics to solve apparently simple problems. More precisely, we will extend the ability to reason deductively in algebra using geometrical constructions. Students will be exposed to critical thinking and technical know-how strategies which they will be expected to master at university, and are transferable to everyday life.

Mark Scheme

Key Skill	1st	2.2	3rd
Numerical Analysis	☐ Simple and complex calculations are correct and always very well presented ☐ Geometrical interpretation is relevant and inform the work	☐ Simple calculations are well attempted and often correct ☐ Formulae from the course material are quoted and often correctly used	☐ Simple calculations are attempted ☐ Formulae from the course material are quoted but not always correctly used ☐ Geometrical interpretation is lacking or largely misguided
Communication	☐ Language is concise, clear and in mathematical style ☐ Appropriate technical vocabulary is confident and correct	☐ Language is clear and reasonably detailed ☐ Appropriate technical vocabulary is not always confident and correct	☐ Attempts are made to use technical vocabulary, but very often incorrect or inappropriate
Critical thinking and contextualisation	☐ The course materials and explanations are coherently applied to new specific situations ☐ Reflective appreciation of the purpose and scope of the course is shown	☐ Some attempts are made to apply course concepts in explaining new problems ☐ Course concepts are applied to familiar problems	☐ Course concepts are quoted but not successfully applied to new problems ☐ Some attempts are made to apply course contents

Glossary of Keywords

C

Continuous function._____

Convergence. The property of approaching a limit, such as a point, line, function or value.

D

Divergence. The property of not converging

I

Interval [a,b]={_____}.

N

Natural number. The whole numbers from 1 upwards: 1, 2, 3, and so on...

Notation. A system of graphic symbols for a specialised use, other than ordinary writing.

R

Real numbers._____

Real-valued function._____

S

Squares number. An integer that is the square of an integer; For example, 9 is a square number, since it can be written as 3×3.

Tutorial 1 – Sums of finite sequences

What is the Purpose of Tutorial 1?

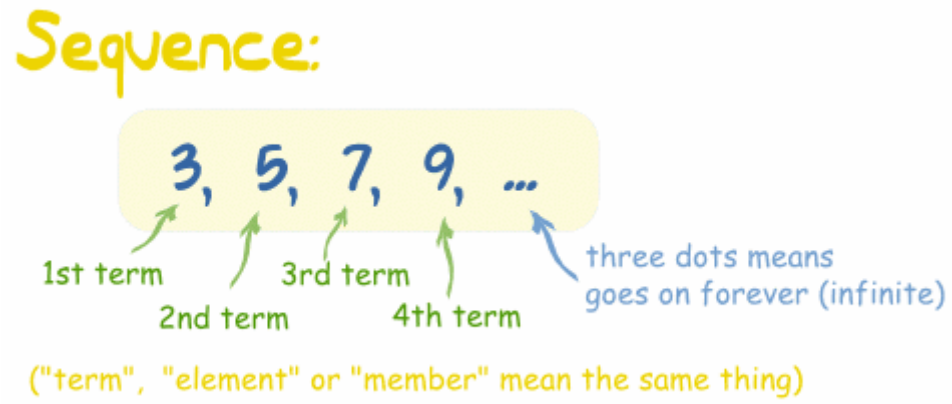
This tutorial will help us to understand the power of formalising real problems and generalise solutions. In particular, we aim to answer the following question. What is the sum of the first 100 natural numbers?

The targets in this tutorial are:

- To introduce the definition of 'finite and infinite sequence'
- To show the difference between finite and infinite sequences
- To learn the notation and the rules of a finite sequence
- To recognise particular finite sequences from their rules
- To introduce the definition of sum
- To work out efficiently the sum of the terms of a given finite sequence.

What is a sequence?

A sequence is a list of things (usually numbers) that are in order.



When the sequence goes on forever it is called an **infinite sequence**, otherwise it is a **finite sequence**.

Examples: (Complete where necessary)

{1, 2, 3, 4, ...} is the sequence of natural numbers (and it is an **infinite sequence**)

{20, 25, 30, 35, ...} is also an **infinite sequence**

{1, 3, 5, 7} is the sequence of the first 4 odd numbers (and is a _____ **sequence**)

{4, 3, 2, 1} is 4 to 1 backwards (_____ **sequence**)

{1, 2, 4, 8, 16, 32, ...} is an _____ **sequence** where every term doubles

{a, b, c, d, e} is the (_____) sequence of the first 5 letters in the alphabet

{0, 1, 0, 1, 0, 1, ...} is the sequence of **alternating** 0s and 1s (note that they are in order; it is an alternating order in this case)

$\left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots\right\}$ is the (_____) sequence of the integer positive powers of $-\frac{1}{2}$

When we say the terms are "**in order**", we are free to define **what order that is**! They could go forwards, backwards or they could alternate, or any type of order we want! A sequence is like a set, except that:

- the terms are **in order** (with sets the order does not matter)
- the same value can appear many times (only once in sets)

Example: {0, 1, 0, 1, 0, 1, ...} is the **sequence** of alternating 0s and 1s. The **set** is just {0,1}.

Notation

Sequences also use the same **notation** as sets:
list each element, separated by a comma, and then put curly brackets around the whole thing.

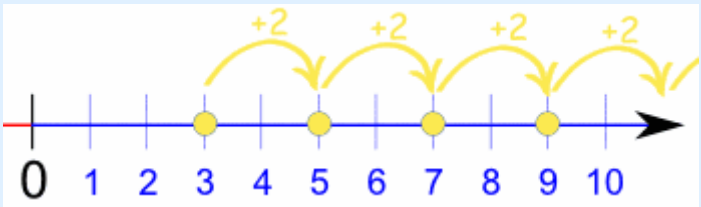
{3, 5, 7, ...}

The curly brackets { } are sometimes called "set brackets" or "braces".

The rule of a sequence

A sequence usually has a **rule**, which is a way to find the value of each term.

Example: the sequence {3, 5, 7, 9, ...} starts at 3 and jumps 2 every time:



Saying "*starts at 3 and jumps 2 every time*" is fine, but it doesn't help us calculate the 10th term, the 100th term, or the nth term, where n could be any term number we want. We want a *formula* with "n" in it (where n is any term number).

So, what can a rule for {3, 5, 7, 9, ...} be?

Firstly, we can see the sequence goes up 2 every time, so we can guess that a Rule is something like "2 times n" (where "n" is the term number). Let's test it out:

Test rule: 2n

n	Term	Test Rule
1	3	$2\mathbf{n} = 2\times\mathbf{1} = \mathbf{2}$
2	5	$2\mathbf{n} = 2\times\mathbf{2} = \mathbf{4}$
3	7	$2\mathbf{n} = 2\times\mathbf{3} = \mathbf{6}$

That nearly worked, so let us try changing it to:

Test rule: 2n+1

n	Term	Test Rule
1	3	$2\mathbf{n}+1 = 2\times\mathbf{1} + 1 = 3$
2	5	$2\mathbf{n}+1 = 2\times\mathbf{2} + 1 = 5$
3	7	$2\mathbf{n}+1 = 2\times\mathbf{3} + 1 = 7$

That works! So instead of saying "starts at 3 and jumps 2 every time" we write this:

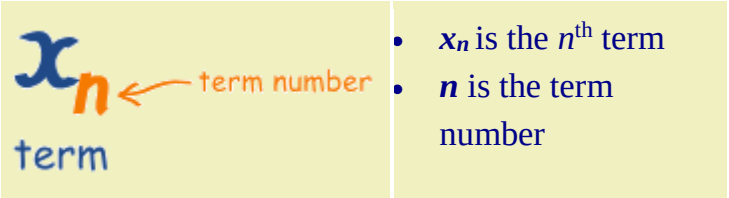
2n+1

Now we can calculate, for example, the **100th term**:

$$2 \times 100 + 1 = \mathbf{201}$$

Notation

To make it easier to use rules, we often use this special way of writing:



Example: to mention the "5th term" we write "x_5"

So a rule for **{3, 5, 7, 9, ...}** can be written like: $x_n = 2n+1$.

To calculate the 10th term we can write:

$$x_{10} = 2\times\mathbf{10}+1 = 21$$

Can you calculate x_{50} (the 50th term) doing this? x_{50}=_____.

Here is another example:

Example: Calculate the first 4 terms of this sequence

$$\{a_n\} = \{ (-1/n)^n \}$$

- $a_1 = (-1/1)^1 = -1$
- $a_2 = (-1/2)^2 = 1/4$
- $a_3 = (-1/3)^3 = -1/27$
- $a_4 = (-1/4)^4 = 1/256$

and so $\{a_n\} = \{ -1, 1/4, -1/27, 1/256, \ldots \}$.

Activity 1. Calculate the first 5 terms of the following sequences:

$\{n\} = \{ _, _, _, _, _, \ldots \}$ and $\{1\} = \{ _, _, _, _, _, \ldots \}$

$$\left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{-\frac{1}{2}, \qquad\qquad\qquad\qquad , \ldots\right\}$$

$$\left\{\frac{n-1}{n}\right\} = \{0, \qquad\qquad\qquad , \ldots\}$$

$$\{(-1)^{n-1}\} = \{1, \qquad\qquad\qquad , \ldots\}$$

$$\left\{\left(\frac{n^2}{2^n}\right)\right\} = \left\{\frac{1}{2}, \qquad\qquad\qquad , \ldots\right\}$$

$$\left\{\left(1+\frac{1}{n}\right)^n\right\} = \left\{2, \left(\frac{3}{2}\right)^2, \qquad\qquad\qquad , \ldots\right\}$$

$$\left\{\left(\frac{\cos(n\pi/2)}{n}\right)\right\} = \left\{0, -\frac{1}{2}, \qquad\qquad\qquad , \ldots\right\}$$

Let's look now at some special sequences, and their rules.

Natural Numbers

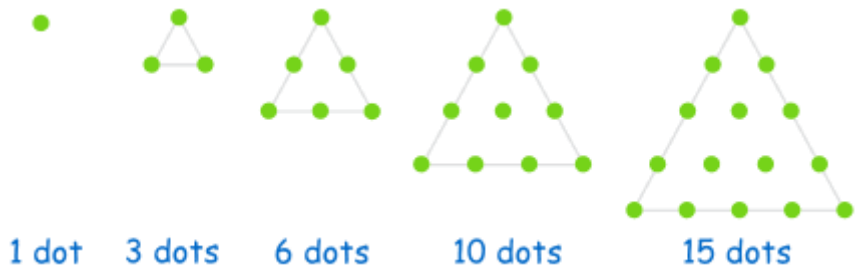
1, 2, 3, 4, 5, 6, 7, 8, 9, ...

The whole numbers from 1 upwards. Rule: $\mathbf{x_n = n}$

Triangular Numbers

1, 3, 6, 10, 15, 21, 28, 36, 45, ...

The Triangular Number Sequence is generated from a pattern of dots which form a triangle. By adding another row of dots and counting all the dots we can find the next number of the sequence:



But it is easier to use the Rule: $\mathbf{x_n = n(n+1)/2}$

Example:

- the 5th Triangular Number is $x_5 = 5(5+1)/2 =$ **15**,
- and the 6th is $x_6 = 6(6+1)/2 =$ **21**

Square Numbers

1, 4, 9, 16, 25, 36, 49, 64, 81, ...

The next number is made by squaring where it is in the pattern. Rule: $\mathbf{x_n = n^2}$

Cube Numbers

1, 8, 27, 64, 125, 216, 343, 512, 729, ...

The next number is made by cubing where it is in the pattern. Rule: $\mathbf{x_n = n^3}$

Series and Partial Sums

In this paragraph we are going to learn to sum up the terms of some special sequences.

Given a **finite** sequence $\{a_n\}_{n=1}^{k}$ we define

$$\sum_{n=1}^{k} a_n \coloneqq a_1 + a_2 + \cdots + a_k$$

the <u>partial (finite) *sum*</u> of the terms of the sequence.

Given an **infinite** sequence $\{a_n\}_{n=1}^{\infty}$ we define

$$\sum_{n=1}^{\infty} a_n \coloneqq a_1 + a_2 + a_3 + \cdots$$

the <u>*series*</u> of the terms of the sequence.

Language Note: Partial Sums are sometimes called **"Finite Series"**
(a **"Series"** is the sum of an infinite sequence).

Let's us focus only on the finite sequences for now.

Partial Sums are often written using the symbol **Σ** (called Sigma) to mean "add them all up".

Sum What?

Sum whatever appears after the Sigma: $\sum n$ (so we sum *n)*

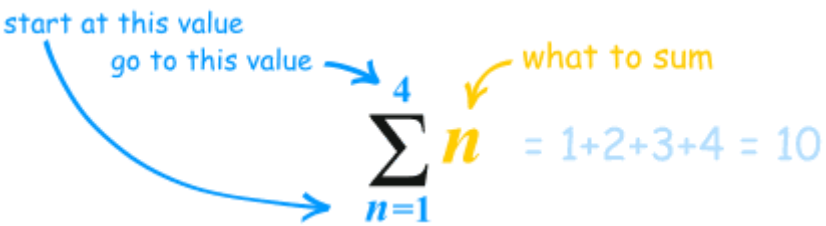
But What is the Value of ***n*** ?

The values are shown below and above the Sigma: $\sum_{n=1}^{4} n$ (it says *n* goes from 1 to 4, which is **1**, **2**, **3** and **4)**

Then?

We add up 1,2,3 and 4: $\sum_{n=1}^{4} n = 1 + 2 + 3 + 4 = 10$

To recap:



More…

Square *n* each time and sum the result:

$$\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

Sum up the first four terms in the sequence **2n+1**":

$$\sum_{n=1}^{4}(2n+1) = 3+5+7+9 = 24$$

We don't have to use *n every time*. In the following example we use "i" and sum up i × (i+1), going from **1** to **3**:

$$\sum_{i=1}^{3} i(i+1) = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 = 20$$

And we can start and end with any number. Here we go from **3** to **5**:

$$\sum_{i=3}^{5} \frac{i}{i+1} = \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$$

Some particular sums

<u>Questions</u>: what is the sum of the first 100 whole numbers??

$$\sum_{n=1}^{k} n = ?$$

How would you work this out efficiently?

There is a well-known story about Karl Friedrich Gauss when he was in elementary school. His teacher got mad at the class and told them to add the numbers from 1 to 100. The question was assigned as "busy work" by the teacher, but Gauss found the answer rather quickly by discovering a pattern. His observation was as follows:

1 + 2 + 3 + 4 + … + 98 + 99 + 100

Gauss noticed that if he was to split the numbers into two groups (1 to 50 and 51 to 100), he could add them together vertically to get a sum of 101.

1 + 2 + 3 + 4 + 5 + … + 48 + 49 + 50

100 + 99 + 98 + 97 + 96 + … + 53 + 52 + 51

1 + 100 = 101
2 + 99 = 101
3 + 98 = 101
.
.
.
48 + 53 = 101
49 + 52 = 101
50 + 51 = 101

Gauss realised that his final total would be 50 x 101 = 5050.

Question. Can you recognise the formula behind this calculation? In other words, how can 5050 be written in terms of 100?

Question. What do you do if there is an odd number of terms to add so that you can't split them into two groups? For example: "how do you work out the sum of the first 21 whole numbers?"

Tutorial 1 – Baseline Test

The homework assignment for the first tutorial is a baseline test to see your initial level of attainment in this subject area. The assignment will test for some or all of the subject specific skills that are required later in the final assignment. However, it is shorter than the final assignment and is an introduction to the subject as well as a challenge!

Do not worry too much about doing 'well' or 'badly' on the baseline test, it takes into account the fact that you may not be familiar with the subject area. It is designed to help you and your PhD tutor identify where you are at the start of the programme and to help you measure your progress along the way.

EXE_1.1 Give the definition of the commutative and associative property of the addition.

EXE_1.2 Find the solution(s) of the following equations and do the check.

$-2 - 13.8n = -8n - (6n + 1)$; $B^2=49$; $Y^3=27$; $(2x+3)^2-2x(x+3)=5x-2(1-x)x$;

$\frac{1}{x^2-9} + \frac{2}{x-3} = \frac{3}{x+3}$; $2x^2 + 5x + 2 = 0$; $x^2 - 6x + 5 = 0$;

EXE_1.3 Multiply the polynomials $(-t^3 + 3t)$ and $(-3t^2 + 2t)$.
Factorize $7p^3r - 5t^3s - 21p^2r - 10\,s^3t^3$.

EXE_1.4

What is the twenty-second term of the sequence given by $x_n = 5n - 3$?

If n is a natural number, what is the tenth term of the sequence given by $x_n = 0.4\times(0.5)^n$?

What is the seventh term of the sequence $\{a_n\} = \{(1/n)^{n-1}\}$?

The nth hexagonal number is given by $h_n = n(2n - 1)$. What is the fifth hexagonal number?

The nth term of a sequence is given by $x_n = 3n^2 - 1$. Which term of the sequence is equal to 866?

The nth term of a sequence is given by $x_n = \frac{n^2-1}{7n-25}$. Which term of the sequence is equal to 2?

Tutorial 2 – Particular finite sums and their properties

What is the Purpose of Tutorial 2?

The purpose of Tutorial 2 is to learn how to sum up efficiently the terms of a finite sequences of numbers. In particular, we will try to answer the following questions.

- What is the sum of the first n natural numbers? (Gauss' formula)
- What is the sum of the first n squares numbers? And cubes?
- What properties does a partial sum have?

Some particular sums

$$\sum_{k=1}^{n} 1 = n$$

Easy! Summing **1, n-times,** *equals* **n.**

$$\sum_{k=1}^{n} c = nc$$

Easy! Summing the constant **c, n-times,** *equals* **c** times **n**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad (2.1)$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad (2.2)$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad (2.3)$$

Let's prove (2.1).

$\sum_{k=1}^{n} k$	=	1	+	2	+	3	+	… +	(n-1)	+	n
$\sum_{k=1}^{n} k$	=	n	+	(n-1)	+	(n-2)	+	… +	2	+	1
$2\sum_{k=1}^{n} k$	=	(n+1)	+	(n+1)	+	(n+1)	+	… +	(n+1)	+	(n+1) = n (n+1)

Then I get that $2\sum_{k=1}^{n} k$= n(n+1) and so $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

To prove (2.2) let us write n-copies of the identity

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1,$$

one for every value of "i" from 1 to n, and let sum them up together.

$$\begin{aligned}
2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\
3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\
4^3 - 3^3 &= 3 \cdot 3^2 + 3 \cdot 3 + 1 \\
\vdots \quad & \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
n^3 - (n-1)^3 &= 3 \cdot (n-1)^2 + 3 \cdot (n-1) + 1 \\
(n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1
\end{aligned}$$

$$\begin{aligned}
(n+1)^3 - 1 &= 3 \cdot \left(\sum_{k=1}^{n} k^2\right) + 3 \cdot \left(\sum_{k=1}^{n} k\right) + n \\
&= 3 \cdot \left(\sum_{k=1}^{n} k^2\right) + 3\,\frac{n(n+1)}{2} + n
\end{aligned}$$

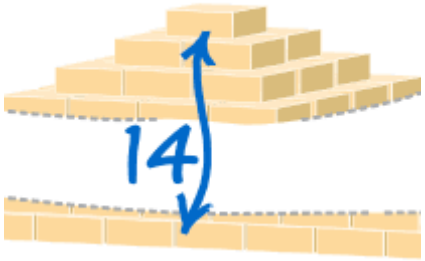
The equation can be solved with respect to $\sum_{k=1}^{n} k^2$ (note that, in the last row, we have used (2.1)).

Moreover, note the cancellation which occurred when we summed the left members of the n identities. The term 2^3 in the first row has cancelled the term -2^3 in the second row, and so on, leaving only two terms $(n+1)^3$ on the nth row and -1^3 from the first row.

$$\sum_{i=1}^{n}((i+1)^3 - i^3) = (n+1)^3 - 1^3.$$

The latter is a **telescopic sum.**

Activity 2: We have the pyramid, as shown in the picture below, made of blocks. There are 14 layers of blocks. How many blocks are in there?



<u>Question.</u> What would do you do to prove (2.3)?

Properties

Partial Sums have some useful properties that can help us do the calculations.

Let $\{a_k\}_{k=\mathrm{m}}^{n}$ be a (finite) sequence of (real) numbers.

Multiplying by a Constant

Let c be a constant.

Then,

$$\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k$$

In other words: if every term we are summing is multiplied by a constant, we can "pull" the constant outside the sigma.

Example:

$$\sum_{k=m}^{n} 6k^2 = 6\sum_{k=m}^{n} k^2$$

In words, instead of summing **$6k^2$** we can sum **k^2** and then multiply the whole result by **6.**

Adding or Subtracting

Let $\{b_k\}_{k=\mathrm{m}}^{n}$ be a (finite) sequence of (real) numbers.

$$\sum_{k=m}^{n}(a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

In words, when two terms are added together, and we want to sum them up, we can actually sum them **separately** and then **add** the results.

Example:

$$\sum_{k=m}^{n}(k + k^2) = \sum_{k=m}^{n} k + \sum_{k=m}^{n} k^2$$

It is going to be easier to do the two sums and then add them at the end.

Note that this property also works for subtraction:

$$\sum_{k=m}^{n}(a_k - b_k) = \sum_{k=m}^{n} a_k - \sum_{k=m}^{n} b_k$$

Activity 3: Work out the following sum

$$\sum_{\mathbf{k=1}}^{\mathbf{n}}(\mathbf{6k^2 - 4k + 3})$$

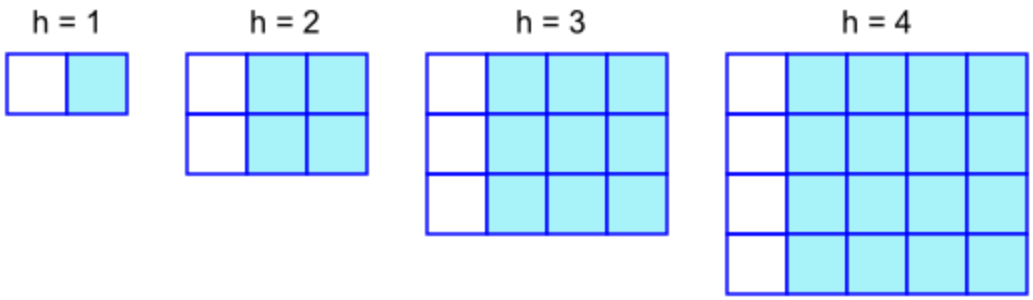
Then, calculate

$$\sum_{\mathbf{k=1}}^{\mathbf{14}}(\mathbf{6k^2 - 4k + 3})$$

Tutorial 2 – homework

EXE_2.1: What is the sum of the whole numbers from 10 to 100 inclusive?

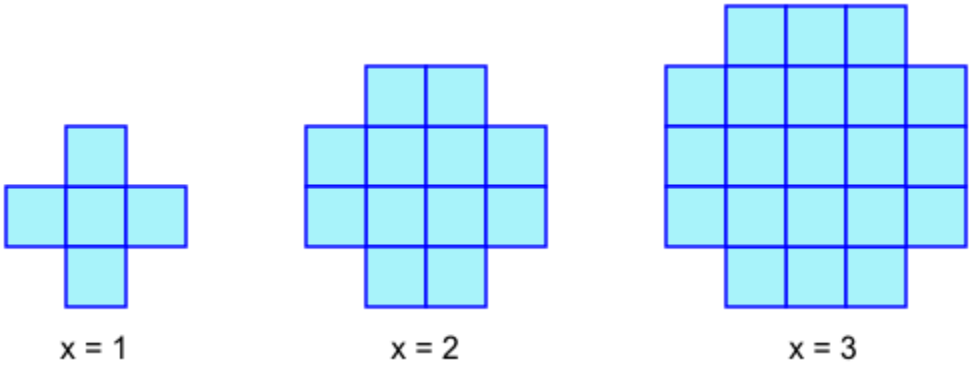
EXE_2.2: The diagram below shows a pattern of rectangles each made up of squares:



Write down a formula for the total number of squares in the first N rectangles (including white and blue squares).

What if N=70?

EXE_2.3: The diagram below shows a pattern of shapes consisting of large squares with four smaller squares cut out from the four corners:



Write down a formula for the total number of small squares in the first N of these shapes.

EXE_3.3 Work out the following telescopic sum.

$$\sum_{n=1}^{N} \frac{1}{n(n+1)}$$

<u>Hint:</u>

$$\frac{1}{n(n+1)} = \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Tutorial 3 – Infinite sequences and their series

What is the Purpose of Tutorial 3?

In Tutorial 3 we will learn that, when dealing with *infinite* sums of numbers, things can get really strange and, sometimes, they can even go against what our mind would initially suggest. In particular, we try to answer the following questions.

- Can I calculate the sum of an infinite sequence of numbers?
- Which are the conditions for this to happen?

Let's say we have an infinite sequence of values

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

which follow a rule (in this case each term is half the previous one).

And we add them **all** up:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = S$$

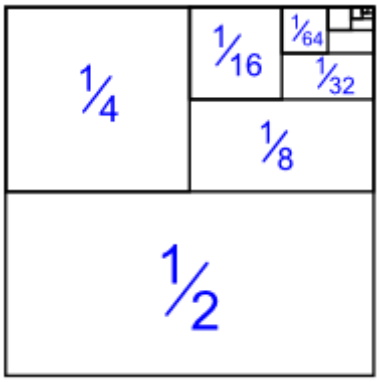
We get an **infinite series**.

"Series" sounds like it is the **list of numbers**, but it is actually when we **add them up**. *(Note: The dots "..." mean "continuing on indefinitely")*

You might think it is impossible to work out the value of S, but in this case we have that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$$

And here is why:



Later we also show a proof using Algebra.

Notation

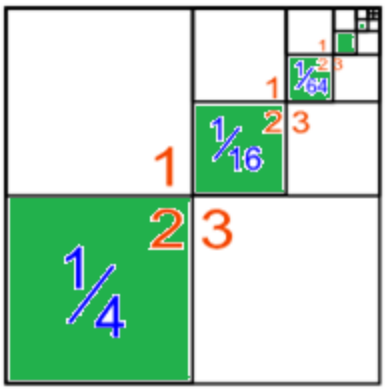
We use Sigma Notation for infinite series with the symbol of infinity ∞ . Our example from above looks like:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

Another Example

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots = \frac{1}{3}$$

Each term is a quarter of the previous one, and the sum equals 1/3:



Of the **3** spaces **(1,2,3)** only **1** gets filled up, hence **1**/**3**.

Converge

Let's add the terms one at a time. When the "sum so far" approaches a finite value, the series is said to be "**convergent**":

Our first example:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Would add up like this:

Term	Sum so far
1/2	0,5
1/4	0,75
1/8	0,875
1/16	0,9375
1/32	0,96875
...	...

The sums are heading towards 1, so this series is **convergent**.
In Calculus we would say "the sequence of partial sums has a **finite limit.**"

Diverge

If the sums do not converge, the series is said to **diverge**.

Example:

$$1 + 2 + 3 + 4 + \ldots$$

Adds up like this:

Term	Sum so far
1	1
2	3
3	6
4	10
5	15
...	...

The sums are just getting larger and larger, not heading to any finite value.

It does not converge, so the series is **divergent**.

Example:

$$1 - 1 + 1 - 1 + 1 \ldots$$

It goes up and down without settling towards some value, so it **diverges**.

More Examples

Geometric Series

A series it is called a **geometric series** when the **ratio** between each term and the next is a constant.

Our first example is a geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

The ratio between each term is **½**. We can show why that series equals 1 using algebra.

First, we will call the whole sum **"S"**: S = 1/2 + 1/4 + 1/8 + 1/16 + ...

Multiply **S** by **½**: S/2 = 1/4 + 1/8 + 1/16 + 1/32 + ...

Now subtract the members! S - S/2 = 1/2

Simplify: S/2 = 1/2

Hence: S = 1

Harmonic Series

This is the Harmonic Series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

It is divergent. How do we know? Let's compare it to another series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots$$

etc...

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \ldots$$

In each case, the top values are equal or greater than the bottom ones.

Now, let's add up the bottom groups:

$$1 + \frac{1}{2} + \frac{1}{4}+\frac{1}{4} + \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} + \frac{1}{16}\ldots + \ldots$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty$$

So our original series $\sum_{n=1}^{k} \frac{1}{n}$ must also be infinite.

Geometric Sequences

In a **Geometric Sequence** each term is found by **multiplying** the previous term by a **constant**.

Example:

2, 4, 8, 16, 32, 64, 128, 256, ...

This sequence has a factor of 2 between each number. Each term (except the first term) is found by **multiplying** the previous term by **2**.

In general, we write a Geometric Sequence like this:

$$\{a, ar, ar^2, ar^3, \ldots\}$$

where:

- **a** is the first term, and
- **r** is the factor between the terms (called the **"common ratio"**)

Example: {1,2,4,8,...}

The sequence starts at 1 and doubles each time, so

- **a=1** (the first term)
- **r=2** (the "common ratio" between terms is a doubling)

And we get: $\{a, ar, ar^2, ar^3, \ldots\} = \{1, 1\times2, 1\times2^2, 1\times2^3, \ldots\} = \{1, 2, 4, 8, \ldots\}$

Note that, r must not be 0. In fact, when r=0, we get the sequence {a,0,0,...} which is not geometric.

The Rule

We can also calculate **any term** using the Rule:

$$x_n = ar^{(n-1)}$$

(we use "n-1" because ar^0 is for the first term).

Example:

10, 30, 90, 270, 810, 2430, ...

This sequence has a factor of 3 between each number. The values of **a** and **r** are:

- **a = 10** (the first term)
- **r = 3** (the "common ratio")

The Rule for any term is:

$$x_n = 10 \times 3^{(n-1)}$$

So, the **4th** term is: $x_4 = 10 \times 3^{(4-1)} = 10 \times 3^3 = 10 \times 27 = 270$

And the **10th** term is: x_{10} = _____

A Geometric Sequence can also have **smaller and smaller** values:

Example:

4, 2, 1, 0,5, 0,25, ...

This sequence has a factor of 0,5 (a half) between each number. Its Rule is

$$\mathbf{x_n = 4 \times (0{,}5)^{n-1}}$$

Why "Geometric" Sequence?

Because it is like increasing the dimensions in **geometry**:

a line is 1-dimensional and has a length of r

in 2 dimensions a square has an area of r^2

in 3 dimensions a cube has volume r^3

etc (yes we can have 4 and more dimensions in mathematics).

Summing a Geometric Series

When we need to sum a Geometric Sequence

$$a + ar + ar^2 + \ldots + ar^{(n-1)}$$

(each term is ar^k, where k starts at 0 and goes up to n-1), there is a handy formula.

$$\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

a is the first term, **r** is the **"common ratio"** between terms, **n** is the number of terms.

The formula is easy to use. Just "plug in" the values of **a**, **r** and **n.**

Example: Sum the first 4 terms of

10, 30, 90, 270, 810, 2430, ...

This sequence has a factor of 3 between each number. The values of **a**, **r** and **n** are:

- **a = 10** (the first term)
- **r = 3** (the "common ratio")
- **n = 4** (we want to sum the first 4 terms)

$$\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

becomes

$$\sum_{k=0}^{4-1}(10 \cdot 3^k) = 10\left(\frac{1-3^4}{1-3}\right) = 400$$

You can check it yourself:

10 + 30 + 90 + 270 = 400

And another example, this time with **r** less than 1:

Activity 4: Add up the first 10 terms of the Geometric Sequence that halves each time:

{ 1/2, 1/4, 1/8, 1/16, ... }

The values of **a**, **r** and **n** are:

- **a =**
- **r =**
- **n =**

So:

$$\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

becomes

<u>Question</u>: *if we continue to increase n, what happens?*

Let's see **why** the formula works, because we get to use an interesting "trick" which is worth knowing.

First, call the whole sum **"S"**: $S = a + ar + ar^2 + \ldots + ar^{(n-2)} + ar^{(n-1)}$

Multiply **S** by **r**: $S \cdot r = ar + ar^2 + ar^3 + \ldots + ar^{(n-1)} + ar^n$

Now subtract members,

$$\begin{array}{rl} S = & a + ar + ar^2 + \ldots + ar^{n-1} \\ -S \cdot r = & \quad - ar - ar^2 - \ldots - ar^{n-1} - ar^n \\ \hline S - S \cdot r = & a \qquad \text{(all these go !)} \qquad - ar^n \end{array}$$

Then, we get a simple result:

$$S - S \cdot r = a - ar^n$$

Let's rearrange it to find S.

Factor out **S** and **a**: $S(1-r) = a(1-r^n)$

Divide by **(1-r)**: $\mathbf{S = a(1-r^n)/(1-r).}$

And so,

$$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

Infinite Geometric Series

When n goes to infinity and r is less than 1, then r^n goes to **zero** and we get:

$$\sum_{k=0}^{\infty} (ar^k) = a\left(\frac{1}{1-r}\right)$$

NOTE: this does not work when r is 1 or more (or less than -1):

r must be between (but not including) **-1 and 1**

and **r should not be 0** because we get the sequence {a,0,0,...} which isn't geometric

Let's bring back our previous example, and see what happens:

Example: Add up ALL the terms of the Geometric Sequence

{ 1/2, 1/4, 1/8, 1/16, ... }

We have:

- **a = 1/2** (the first term)
- **r = 1/2** (halves each time)

And so:

$$\sum_{k=0}^{\infty}\left(\tfrac{1}{2}\cdot\left(\tfrac{1}{2}\right)^k\right) = \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right)$$

$$= ½ \times 1 / ½ = 1$$

Yes ... adding (1/2)+(1/4)+(1/8)+... equals **exactly 1**.

Application - Recurring Decimal

Does 0,999... equal 1?

Calculate 0,999...

We can write a recurring decimal as a sum like this:

$$\begin{aligned} 0.999... &= 0.9 + 0.09 + 0.009 + ... \\ &= 0.9\cdot 0.1^0 + 0.9\cdot 0.1^1 + 0.9\cdot 0.1^2 + ... \\ &= \sum_{k=0}^{\infty} 0.9\cdot 0.1^k \end{aligned}$$

And now we can use the formula:

$$\sum_{k=0}^{\infty} 0.9\times 0.1^k = 0.9\left(\frac{1}{1-0.1}\right) = 0.9\left(\frac{1}{0.9}\right) = 1$$

Yes! 0,999... **does** equal 1.

There are more types of Infinite Series, and it is interesting (and often challenging!) to work out if they are convergent or not, and what it might converge to.

Alternating Series

An example of an Alternating Series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2)$$

It moves up and down, but in this case converges on the natural logarithm of 2.

Tutorial 3 – homework

EXE_3.1: What is the eleventh term of the geometric sequence 3, 6, 12, 24, ... ?

EXE_3.2: What is the ninth term of the geometric sequence 81, 27, 9, 3, ... ?

EXE_3.3: What is the sum of the first eight terms of the geometric sequence 5, 15, 45, ... ?

Tutorial 4 – Graph of simple functions

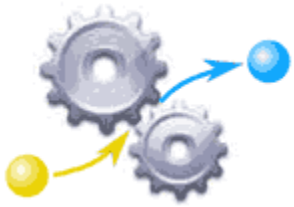
What is the Purpose of Tutorial 4?

In this tutorial we will master basic facts about the graph of a function on the Cartesian product. In particular, we answer the following questions.

- What is a function?
- What is the graph of a function?

What is a Function?

A function relates an input to an output.



It is like a machine that has an input and an output.

And the output is related somehow to the input.

f(x)	"**f(x) = ...** " is the classic way of writing a function. And there are other ways, as you will see!

Input, Relationship, Output

We will see many ways to think about functions, but there are always three main parts:

- The input
- The relationship
- The output

Example: "Multiply by 2" is a very simple function. Here are the three parts:

Input	***Relationship***	**Output**
0	× 2	0
1	× 2	2
7	× 2	14
10	× 2	20
...	...	...

For an input of 50, what is the output?

Some Examples of Functions

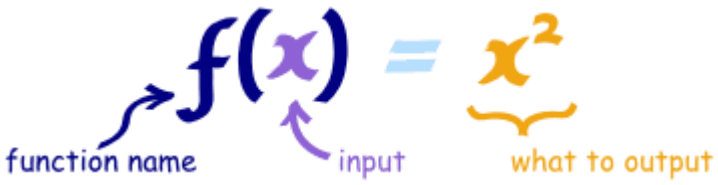
- $\mathbf{x^2}$ (squaring) is a function
- $\mathbf{x^3+1}$ is also a function
- Sine, Cosine and Tangent are functions used in trigonometry
- and there are lots more!

But we are not going to look at specific functions. We will look at the **general idea** of a function.

Names

First, it is useful to give a function a **name**.

The most common name is "***f***", but we can have other names like "***g***".



We say *"f of x equals x squared"*

What goes **into** the function is put inside parentheses () after the name of the function:

So ***f(x)*** shows us the function is called "***f***", and "***x***" goes **in**

And we usually see what a function does with the input:

f(x)* = *x*²** shows us that function "f***" takes "***x***" and squares it.

Example: with $\mathbf{f(x) = x^2}$ an input of 4 becomes an output of 16. So we can write

$$\mathbf{f(4) = 16}$$

Don't get too concerned about "x", it is just there to show us where the input goes and what happens to it. It could be anything!

The function $f(x) = 1 - x + x^2$ is the same function as:

$f(q) = 1 - q + q^2$ $\quad$ $h(A) = 1 - A + A^2$ $\quad$ $w(\theta) = 1 - \theta + \theta^2$

The variable (x, q, A, etc.) is just there so we know where to put the values:

$$f(\mathbf{2}) = 1 - \mathbf{2} + \mathbf{2}^2 = 3$$

Sometimes a function has no name, and we see something like:

$$y = x^2$$

But there is still:

- an input (x)
- a relationship (squaring)
- and an output (y)

At the top we said that a function was **like** a machine. But a function doesn't really have belts or cogs or any moving parts - and it doesn't actually destroy what we put into it!

A function ***relates*** an input to an output.

Saying "**f(4) = 16**" is like saying 4 is somehow related to 16. Or 4 → 16

Example: A tree grows 20 cm every year, so the height of the tree is ***related*** to its age using the function ***h***:

$$\boldsymbol{h}(\mathbf{age}) = \mathbf{age} \times \mathbf{20}$$

So, if the age is 10 years, the height is $\boldsymbol{h}(\mathbf{10}) = \mathbf{10} \times \mathbf{20} = \mathbf{200}$ **cm.**

Here are some example values:

age	h(age) = age × 20
0	0
1	20
3,2	64
15	300
...	...

What Types of Things Do Functions Process?

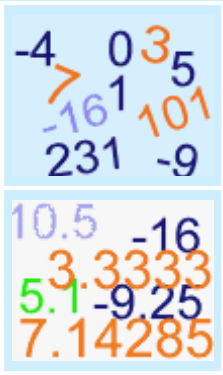
"Numbers" seems an obvious answer, but...

... **which** numbers?

For example, the tree-height function ***h*(age) = age×20** makes no sense for an age less than zero.

... it could also be letters ("A"→"B"), or ID codes ("A6309"→"Pass") or stranger things.

So we need something more powerful, and that is where sets come in.



A set is a collection of things.

Here are some examples:

Set of even numbers: {..., -4, -2, 0, 2, 4, ...}
Set of clothes: {"hat","shirt",...}
Set of prime numbers: {2, 3, 5, 7, 11, 13, 17, ...}
Positive multiples of 3 that are less than 10: {3, 6, 9}

Each individual thing in the set (such as "4" or "hat") is called **element**.

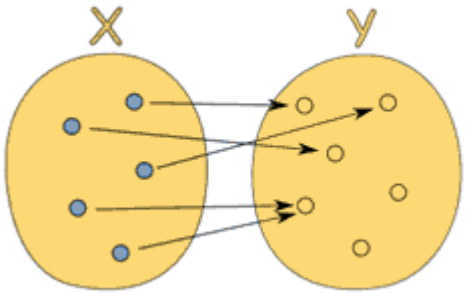
So, a function takes **elements of a set**, and gives back **elements of a set**.

A Function is Special

But a function has **special rules**:

- It must work for **every** possible input value
- And it has only **one relationship** for each input value

This can be said in one definition:



Formal Definition of a Function

A function is a law which relates **each element** of a set
with **exactly one** element of another set
(possibly the same set).

The Two Important Things!

1. "...each element..." means that every element in **X** is related to some element in **Y**.

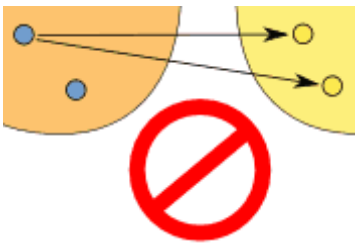
   We say that the function ***covers* X** (relates every element of it).

   Some elements of **Y** might not be related to at all (which is fine).

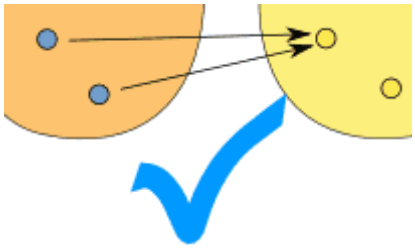
2. "...exactly one..." means that a function is ***single valued***. It will not give back 2 or more results for the same input.

So "f(2) = 7 **or** 9" is not a function!

Note: "One-to-many" is **not** allowed, but "many-to-one" is allowed:



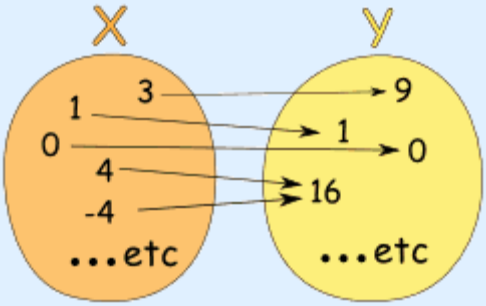
(one-to-many)
This is **NOT** OK in a function



(many-to-one)
But this **is** OK in a function

When a relationship does **not** follow those two rules then it is **not a function**... it is still a **relationship**, just not a function.

Example: The relationship $x \rightarrow x^2$



Could also be written as a table:

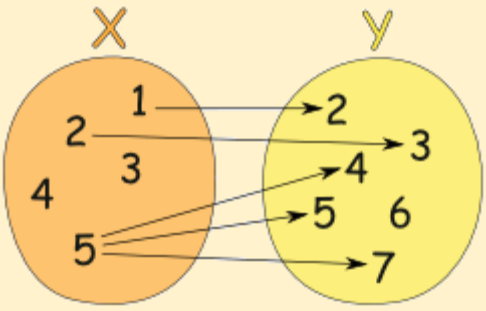
X: x	Y: x^2
3	9
1	1
0	0
4	16
-4	16
...	...

$x \rightarrow x^2$ is a function, because:

- Every element in X is related to Y
- No element in X has two or more related elements in Y

So it follows the rules. Notice how both **4** and **-4** relate to **16**, which is allowed.

Activity 5: Is the following relationship a function?



Infinitely Many

Functions usually work on sets with infinitely many elements.

Example: $y = x^3$

- The input set "X" is the set of all Real Numbers
- The output set "Y" is also the set of all the Real Numbers

We can't show ALL the values, so here are just a few examples:

X: x	Y: x^3
-2	-8
-0,1	-0,001
0	0
1,1	1,331
3	27
and so on...	and so on...

Ordered Pairs

And here is another way to think about functions:

Write the input and output of a function as an "ordered pair".

They are called **ordered** pairs because the input always comes first, and the output second:

(input, output)

So it looks like this:

(**x**, **f(x)**)

Example: If the function takes in "4" and gives out "16" we write **(4, 16)**.

Set of Ordered Pairs

A function can then be defined as a **set** of ordered pairs:

Example: **{(2,4), (3,5), (7,3)}** is a function that says

"2 is related to 4", "3 is related to 5" and "7 is related 3".

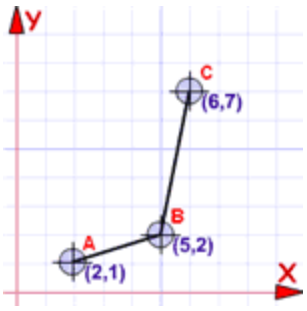
Also, notice that:

- the domain is **{2,3,7}** (the input values)
- and the range is **{4,5,3}** (the output values)

Remember, the function has to be **single valued**, so we also say "if it contains (a, b) and (a, c), then b must equal c", which is just a way of saying that an input of "a" cannot produce two different results.

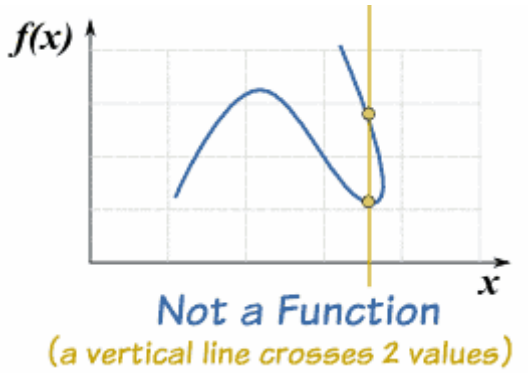
Example: {(**2**,**4**), (**2**,**5**), (7,3)} is **not** a function because {2,4} and {2,5} means that 2 could be related to 4 **or** 5.

In other words it is not a function because it is **not single valued**



We can graph ordered pairs because they are also coordinates!

The set of all ordered pairs (x, f(x)) in the product X x Y is called the **graph** of the function f.



Vertical Line Test

On a graph, the idea of **single valued** means that no vertical line ever crosses more than one value.

If it **crosses more than once** it is still a valid curve, but is **not a function**.

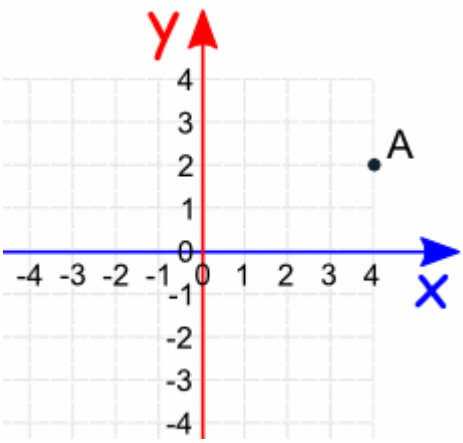
Some types of functions have stricter rules. To find out more you can read Injective, Surjective and Bijective.

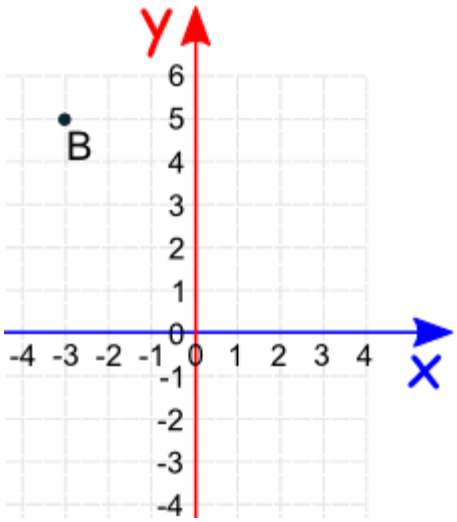
Conclusion

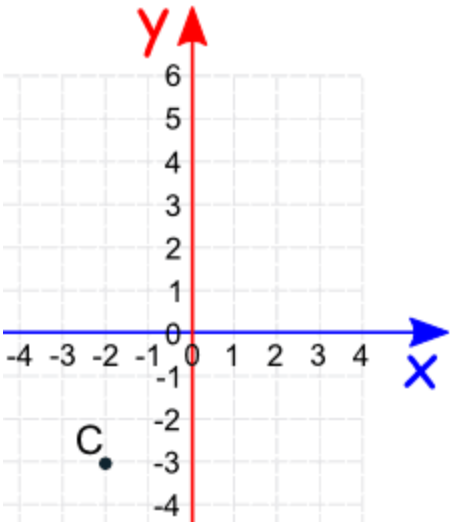
- a function **relates** inputs to outputs,
- a function takes elements from a set (the **domain**) and relates them to elements in a set (the **codomain**),
- all the outputs (the actual values related to) are together called the **range,**
- a function is a **special** type of relation where **every element** in the domain is included, and any input produces **only one output**,
- an input and its matching output are together called an **ordered pair,** so a function can also be seen as a **set of ordered pairs following the rules above.**

Tutorial 4 – Homework: Draft Assignment

EXE_4.1: What are the Cartesian coordinates of the points A, B and C?







EXE_4.2: Consider A = (2, 5), B = (2, -1), C = (-6, -1) and D = (-6, 5).

A is joined to B, B to C, C to D and D to A with straight lines. What shape is ABCD?

(Hint: sketch a graph and put the points on it)

EXE_4.3: Write an equation to represent the function from the following table of values:

x	y
-2	-4
-1	-2
0	0
1	2
2	4

Write an equation to represent the function from the following table of values:

x	y
-2	1
-1	2
0	3
1	4
2	5

EXE_4.4: The function f is defined on the real numbers by $f(x) = 2 + x - x^2$. What is the value of f(-3)?

EXE_4.5: What is the sum of the first nine terms of the geometric sequence 20, 10, 5, ... ?
Give your answer as a decimal correct to 1 decimal place.

EXE_4.6: For the sequence defined by $a_n = n^2 - 5n + 2$, what is the smallest value of n for which a_n is positive?

Tutorial 5 – Areas as limit of sums

What is the Purpose of Tutorial 5?

The purpose of the tutorial is to show how to calculate the area of a region under the graph of a non-negative real-valued function making use of what we have learnt about sums and series.

- How can I calculate the area of a region under a curve?

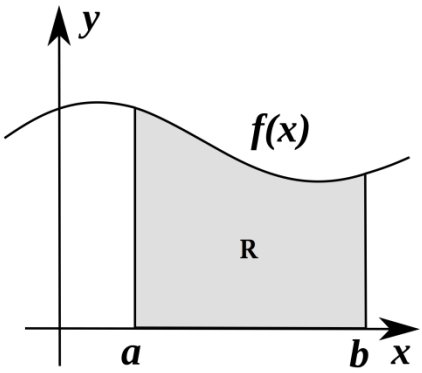
Introduction to integration

Let us suppose we all know what finding the area of a geometric figure means and let us review some of its properties.

- The area of a plan region is a non-negative real number expressed w.r.t. a fixed unity of measurement. In other words, is the number of unit squares contained within the edge of the figure
- The area of a rectangle with width W and height H is A=W x H
- Congruent regions have the same areas
- If the region S in contained in the region R, then the area of S is less equal than the area of R
- If the region R is the union of regions with no overlaps, then the area of R is he sum of the areas of those regions.

Thanks to these properties we are able to calculate the area of any polygon, i.e. regions bounded by parts of a straight line. However, we cannot go beyond polygons without making a passage to a limit.

In this tutorial we consider the problem of finding the area of a region R which is under the graph of a real-valued function f over the x-axis and between two end points "a" and "b" on it with a<b.



In order to do this, we proceed as follows.
We divide the interval [a,b] in n subintervals through the points of division

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$$

Let us call Δx_i the length of the i-th subinterval $[x_i - x_{i-1}]$

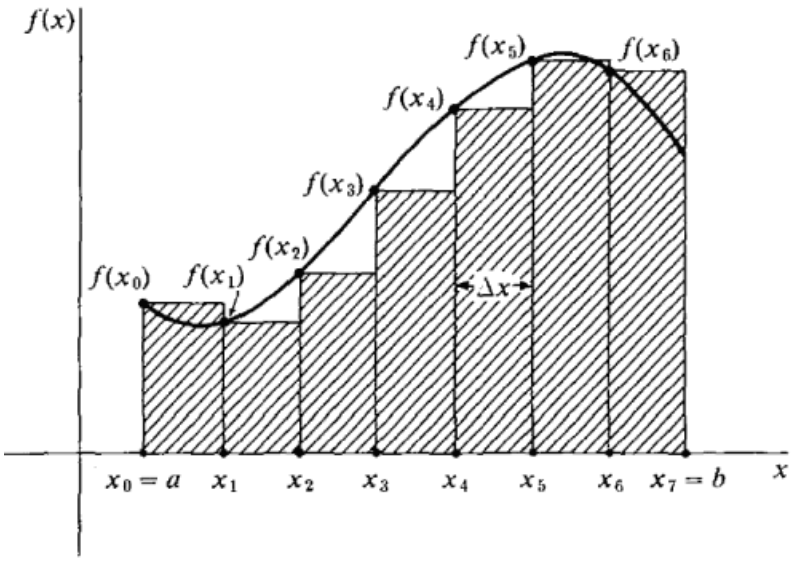
$$\Delta x_i = x_i - x_{i-1}$$

Let us build on every subinterval $[x_i - x_{i-1}]$ a vertical rectangle which has base Δx_i and height $\mathrm{f}(x_i)$.

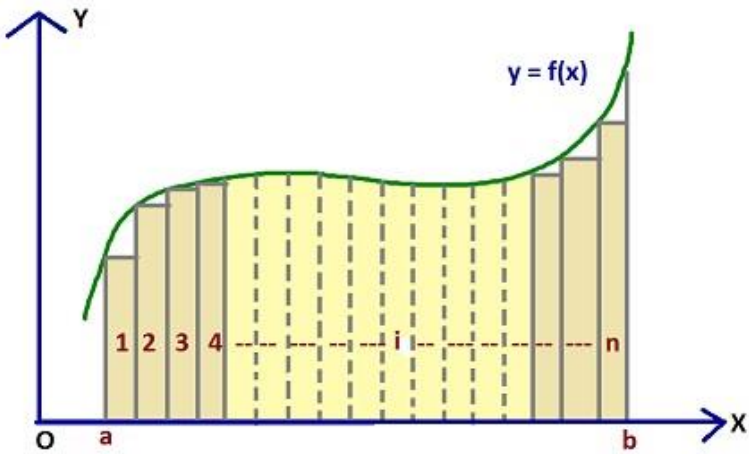
The area of this rectangle is $f(x_i) \cdot \Delta x_i$.

Let us consider now the sum of all these rectangles

$$S_n = f(x_0) \cdot \Delta x_1 + f(x_1) \cdot \Delta x_2 + f(x_2) \cdot \Delta x_3 + \cdots + f(x_{n-1}) \cdot \Delta x_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x_i$$



Obviously, S_n is an approximation of the area of the region R but this approximation become even better if we increase the number of subintervals $[x_i - x_{i-1}]$ so that Δx_i become smaller. Note that the quantity Δx_i represents the thickness of a small but finite rectangle.



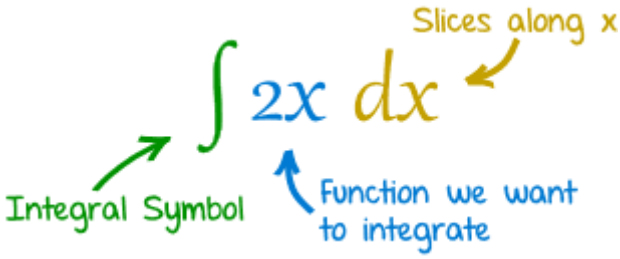
Using a more sophisticated mathematical language, we could say that the area of the region R is seen as the limit of the area S_n when n goes to infinity (Δx_i goes to 0).

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_{i-1}) \Delta x_i .$$

If this sum can actually be found, $\int_a^b f(x)dx$ is called the **definite integral** of f(x), from x = a to x = b.

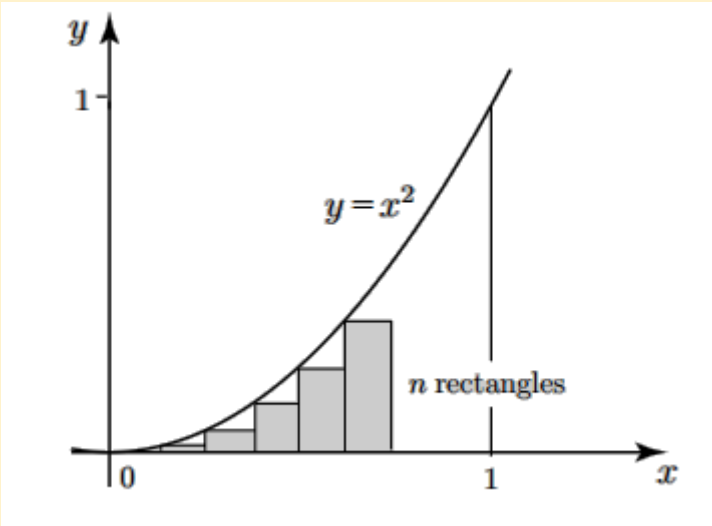
Notation

The symbol for "Integral" is a stylish "S" for "Sum" (summing slices):



After the Integral Symbol we put the function we want to find the integral of (called the Integrand), and then finish with **dx** to mean the slices go in the x direction (and approach zero in width).

Activity 6: Use the idea of the limit of a sum to find the area under the graph of $y = x^2$ between $x = 0$ and $x = 1$.



Refer to the graph in the figure above to help you answer the questions.

The area under $y = x^2$ is approximated by a number of thin rectangles.

(a) If the interval between $x = 0$ and $x = 1$ is divided into n rectangles what is the width of each rectangle? Answer:

(b) What is the x coordinate at the left-hand side of the first rectangle? Answer:

(c) What is the x coordinate at the left-hand side of the second rectangle? Answer:

(d) What is the x coordinate at the left-hand side of the third rectangle? Answer:

(e) What is the x coordinate at the left-hand side of the kth rectangle? Answer:

(f) Given that $y = x^2$, what is the y coordinate at the left-hand side of the kth rectangle? Answer:

(g) The area of the kth rectangle is its height × its width. Write down the area of the kth rectangle. Answer:

(h) Work out the total area S_n of the n rectangles. Answer:

(i) Let n tend to infinity to obtain the exact answer:

Tutorial 5 – Homework: Final Assignment

Mastering basic facts in addition

EXE_5.1

What is the sum of the first six triangular numbers?

What is the sum of the squares of the first eighty natural numbers?

EXE_5.2

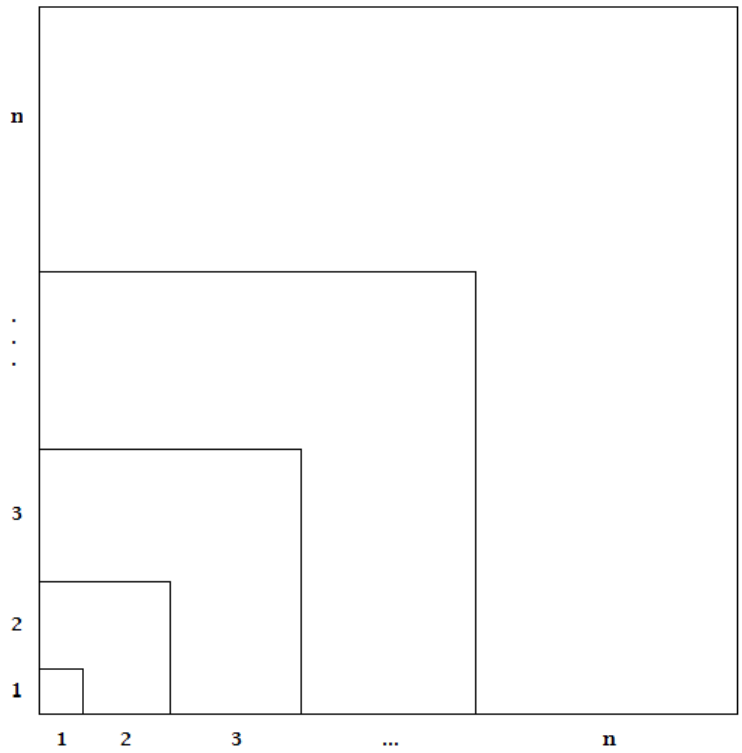
Show that $\sum_{n=1}^{k}(n^2 - n - 1) = \frac{k(k-2)(k+2)}{3}$.

Hence calculate the value of $\sum_{n=10}^{40}(n^2 - n - 1)$.

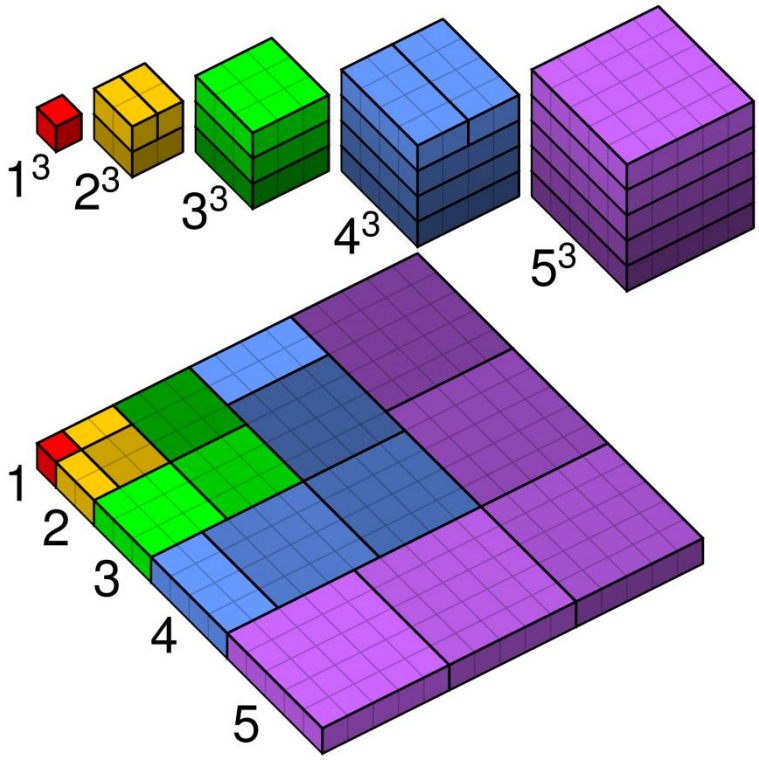
EXE_5.3

The picture below shows a square with edge long $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ divided in a little square of edge 1 and in n-1 regions having an "L" form whose minor edges are long 2,3,…n.

- Show that the area of the "L" regions with minor edge "i" is "i^3"
- Then verify that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.



A geometrical interpretation for n=5 is the following.



EXE_5.4

Add up all the terms of the following infinite geometric sequence:

$$\left\{\frac{1}{3}, \frac{-1}{9}, \frac{1}{27}, \frac{-1}{81}, \ldots\right\}$$

EXE_5.5

Find the integral of the function f(x)=1-x between 0 and 1.

EXE_5.6

Find the integral of the function $f(x)=x^3$ between 0 and $b>0$.

Tutorial 6 – Feedback

What is the Purpose of Tutorial 6?

- Reflect on skills learned on the programme with a view to encouraging resilience
- Receive feedback on final assignment

What are the key questions?

- What strengths have I demonstrated in my work and what areas of development are there left for me to address?
- What steps do I need to take to improve my academic output?

Activity 2 – Marking your own

Using the mark scheme, identify three things you did well and three things you could improve.

What went well?(WWW)

1.

2.

3.

What could be improved? (EBI)

1.

2.

3.

Activity 3 – Responding to feedback

Read through your feedback sheet and highlight on your assignment where you received positive feedback (What Went Well) and areas for improvement (Even Better If). You might find it helpful to annotate this with a brief comment or two.

(Acknowledgement: The structure and the activities in this feedback tutorial were first designed by Hannah Manktelow in January 2015. We have repeated them here as they demonstrate thoughtful and excellent planning, and can be used by all tutors to ensure a successful and productive feedback tutorial.)

University Applications Guidance

For course choices and careers information:	www.brightknowledge.org.uk
For information and statistics on universities and course:	www.unistats.direct.gov.uk
For comprehensive information on universities, including rankings:	www.thecompleteuniversityguide.co.uk

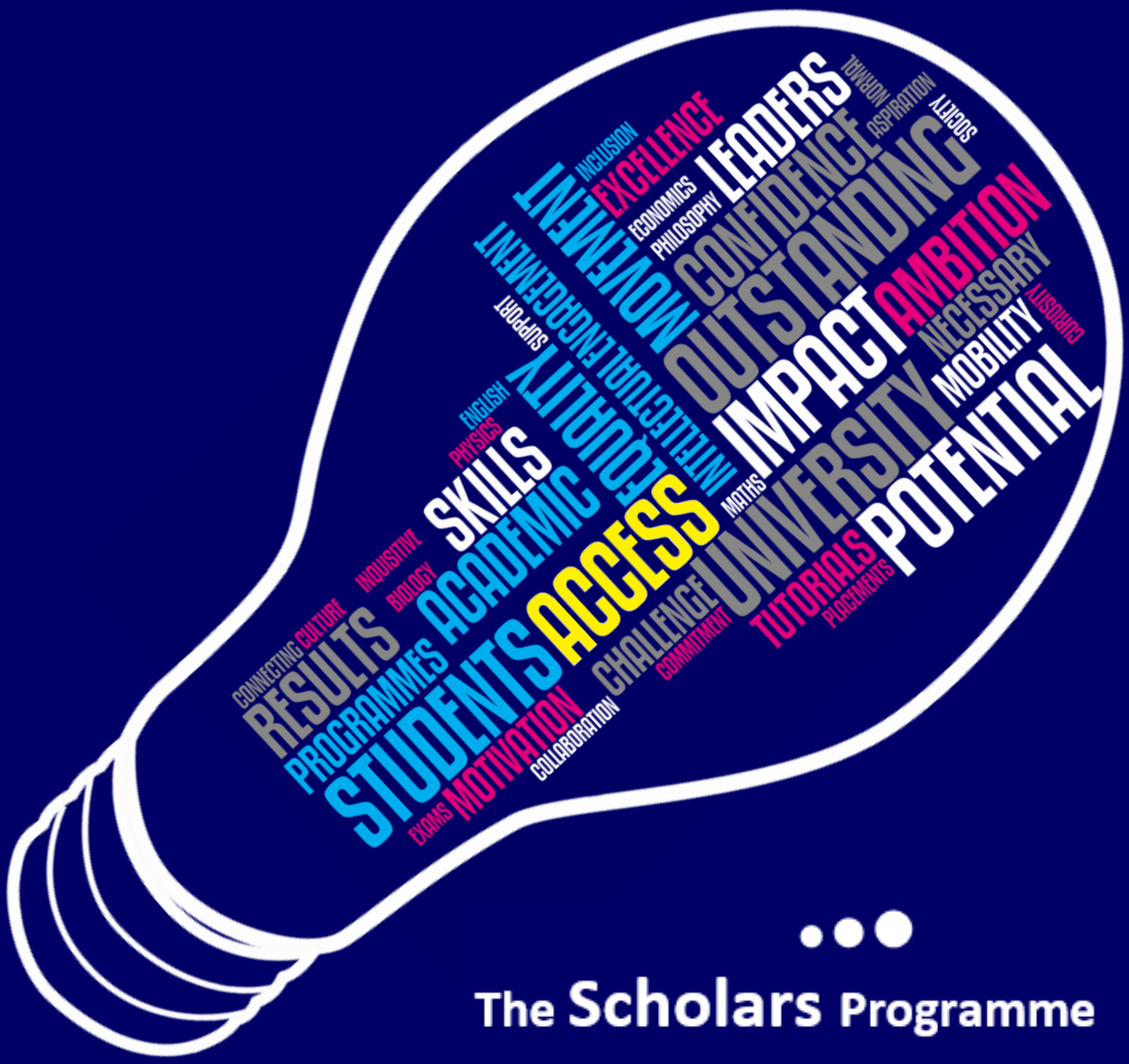
PLEASE NOTE – The Scholars Programme is designed to support the university applications process at your school. The work you complete for the programme should add to your schoolwork rather than detract from it. Please be aware of deadlines and try to manage your workload appropriately. If you are worried about the Scholars Programme interfering with your schoolwork then please speak to your teacher.

Subject Specific Top Tips for Strengthening your application

What could you read around the subject?
www.intmath.com/blog/mathematics/the-twelve-days-of-christmas-how-many-presents-1686
www.intmath.com/applications-integration/applications-integrals-intro.php
Evaluating π and e with series

Suggestions of podcasts/videos that can be found online?
Zeno's paradox on www.numberphile.com or https://www.youtube.com/watch?v=skM37PcZmWE

For course choices and careers information:
www.furthermaths.org.uk/fmincareer and www.mathscareers.org.uk



The Scholars Programme