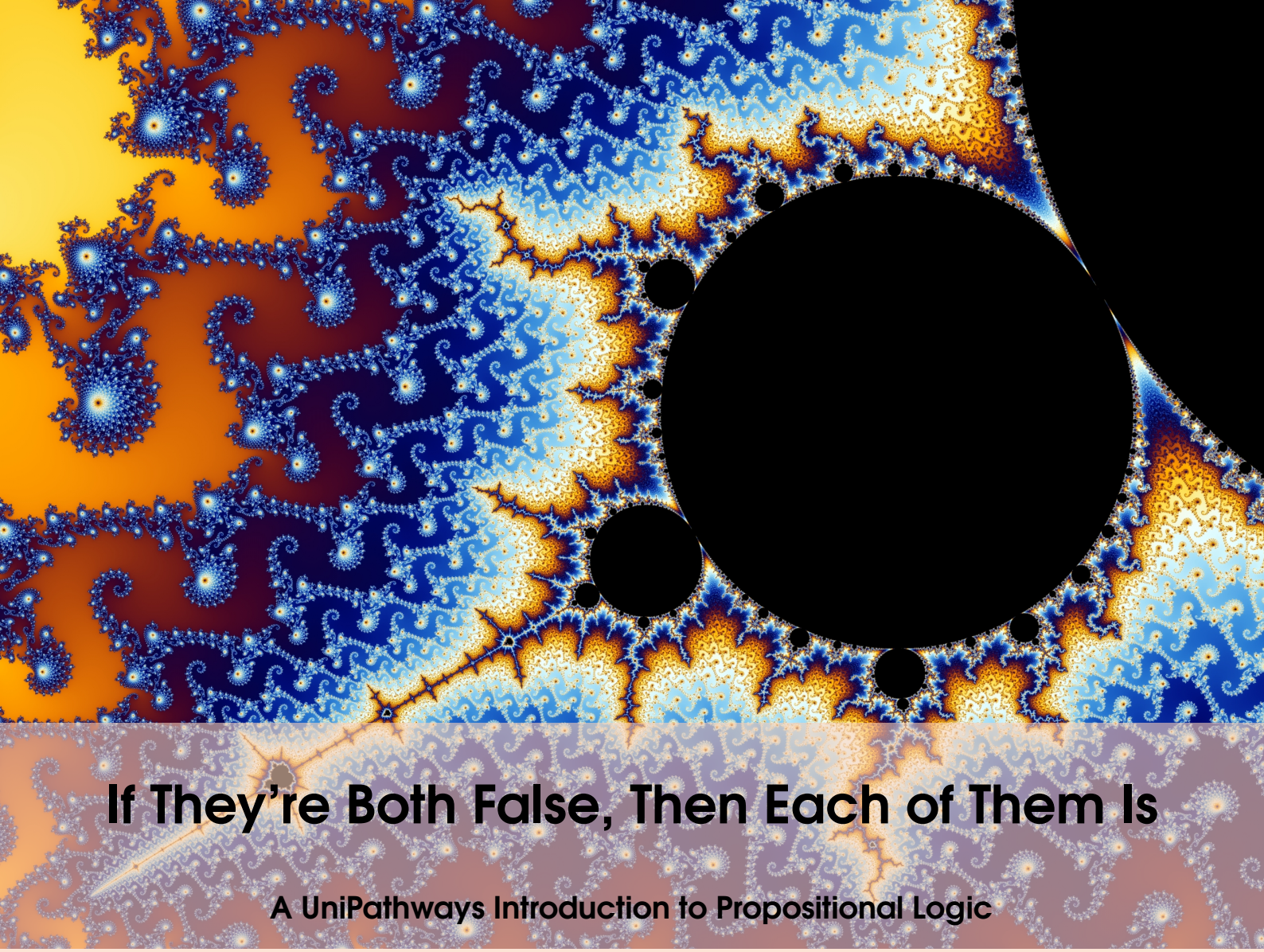

If They're Both False, Then Each of Them Is

A UniPathways Introduction to Propositional Logic

Dr. Nicola Spinelli

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Why this Course and How to get the Most out of it

This is your UniPathways course Handbook for this year. This is stuff you will never see at school (not even at A-Level); in fact, you may not see it even if you go on to do Mathematics at a top university. It is degree material, but – hopefully – pitched just above your current level. So it is meant to stretch you, but also to be accessible for you.

In each chapter there will be exercises at critical points. These correspond to some of the activities we are going to do in the tutorials. There will be space at the end of each chapter to do exercises. Should that not be enough, feel free to use additional paper, but be sure to attach it to the Handbook.

Homework is not included in the Handbook. I will set it at every tutorial and give all the relevant materials. Homework should be done on A4 paper and handed in to me on the agreed day. You will then attach it to your Handbook (there are a few blank pages at the end of each chapter to that end). You will not have a full week to complete your homework, though exactly how long you will have depends on when tutorials will take place within the week.

In every tutorial I will give you some extension materials, which you will then attach to you Handbook. These may be more difficult exercises, additional content, or independent research tasks. Do the extensions only if you feel comfortable with the core. Try to locate how comfortable you feel on a 1-to-10 scale, where 1 is 'I have no idea what I'm doing' and 10 is 'This is so easy, where's my degree?'. If you feel you are 6 or higher, have a go at the extensions.

I want to stress that your Handbook is *yours*: I will never mark it. At university you will be required to take ownership of you learning. The earlier you start experiencing this, and experimenting, the better.

Tutorial 6's homework is your final assignment. You will complete it as if it were normal homework, but it will be marked and graded according to a university-style mark scheme. The specific mark scheme is included in this Handbook. Obviously you should aim for a First (and if you have been selected for this course, believe me: you can make it), but shouldn't be disappointed if you get an Upper Second: that's a pretty good grade too!

It is possible that you will find the style of this Handbook somewhat difficult. This is roughly what a university maths textbook looks like. Don't be scared, though: the actual tutorials and the

additional materials I will provide will make it much more accessible. As you will see, it only takes a little help.

At the very end of the Handbook you will find an index. This is where all the key words may be found. However, you might encounter new words while you work your way through the pages. My advice is to note them down along with a definition (to be found in context or in a dictionary). At the end of each chapter there is space to do so – and for notes of any kind.

Have fun!

Nicola

0.1 Course Rationale

It may sound surprising to pupils that mathematics applies to something else than nature (as in physics) or social phenomena (as in sociology or economics). Yet it does. Mathematical or symbolic logic is an application of mathematics to an important area of reality: reasoning. It translates English sentences in mathematical language and represents how they connect to yield valid arguments.

In this course we will work our way into some of the basics of propositional logic, which is the fundamental level of mathematical logic. We will look at: 1) how to translate arguments from English to the language of propositional logic; 2) how to check their validity by means of truths tables; 3) how to check tautologies by means of truth tables; 4) how to work out mathematical proofs of the laws of propositional logic.

Apart from being an unusual and interesting application of mathematics, symbolic logic is an exceptionally useful tool for developing the ability – invaluable in any university-level course, not to mention life in general – to think clearly, precisely and critically. Being able to reason formally will also be extremely helpful in pupils' mathematical practice – as well as being itself a piece of mathematical practice: propositional logic connects with areas of discrete mathematics such as the algebra of sets and Boolean algebra in general. Pupils will also learn how to work in groups and communicate their findings (or their difficulties) in a university-style small-group teaching and learning environment.

The final assignment will be graded against a university-style mark scheme, and will call on students to use concepts and techniques learnt in the tutorials to prove two pairs of logical laws: the Idempotence Laws and De Morgan's Laws. One of De Morgan's Laws, in particular, says that if two sentences are both false, then each of them is. Sounds trivial, doesn't it? But why believe it if you can't prove it?

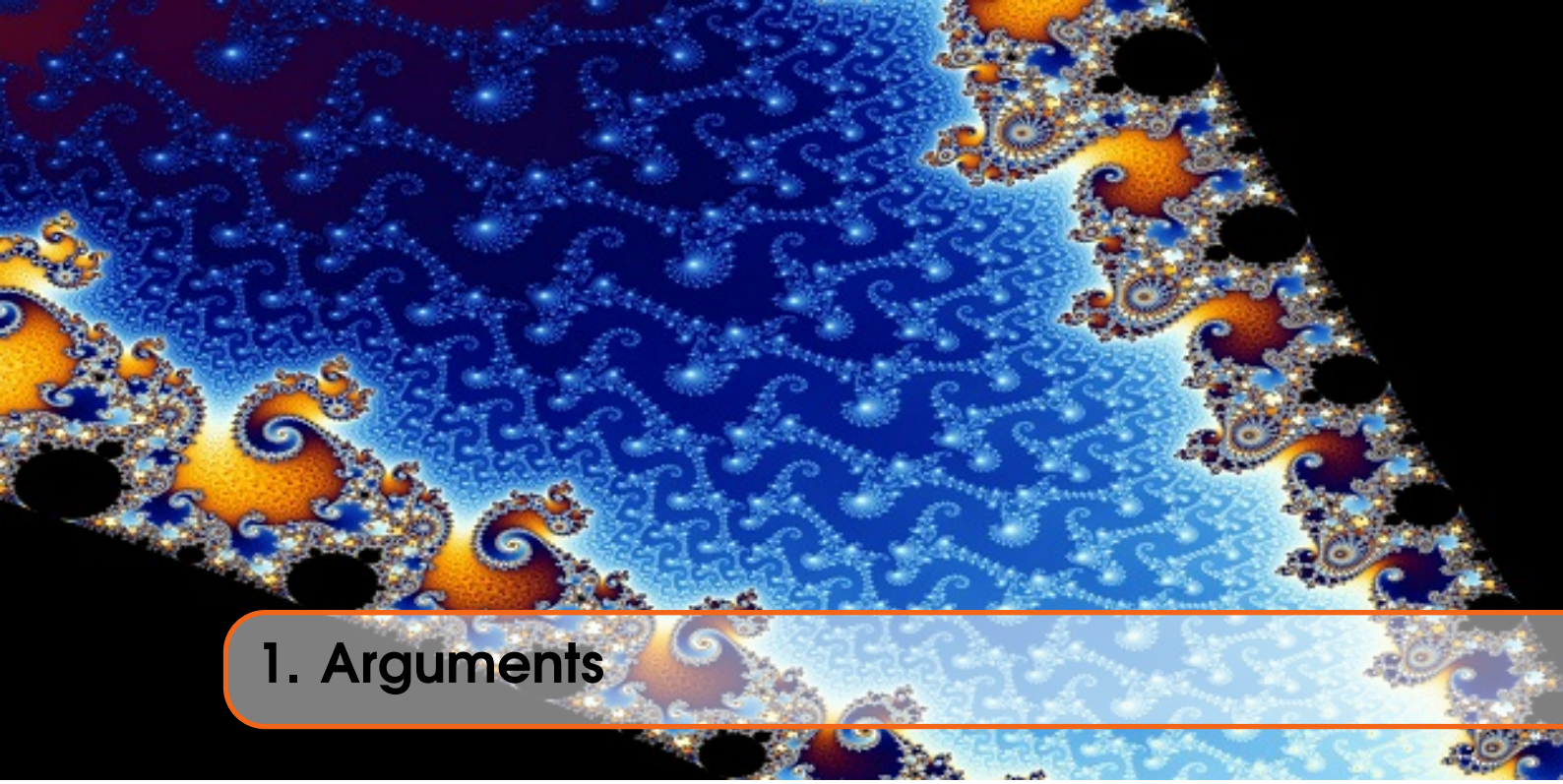
0.2 Mark Scheme

Mark scheme for 1st and 2.1 (or Upper Second)

Key Skill	1st	2.1
Calculation Technique	☐ Algebra of sets: sets are consistently constructed correctly – showing a good understanding of the underlying operations and concepts. ☐ Logic: compound sentences are consistently constructed correctly – showing a good understanding of the syntax of the connectives. Truth tables are consistently constructed correctly – showing a good understanding of the relevant matrix construction techniques.	☐ Algebra of sets: sets are often constructed correctly – showing a fair understanding of the underlying operations and concepts ☐ Logic: compound sentences are often constructed correctly – showing a fair understanding of the syntax of the ☐ connectives. Truth tables are often constructed correctly – showing a fair understanding of the relevant matrix construction techniques.
Proof Technique	☐ Proof strategies are consistently present (implicitly or explicitly) and implemented correctly. Theorems and/or previous results, if invoked, are consistently invoked and deployed correctly. Justification is given at every step and is consistently correct. Proofs are consistently presented in a tidy and accessible way.	☐ Proof strategies are often present (implicitly or explicitly) and often implemented correctly. Theorems and/or previous results, if invoked, are often invoked and deployed correctly. Justification is given at every step and is often correct. Proofs are consistently presented in a tidy and accessible way.
Language	☐ Specialist vocabulary is consistently and correctly used.	☐ Specialist vocabulary is sometimes used, and not always correctly.
Creativity	☐ Proofs, as well as calculations, often show intellectual initiative and insights beyond the content of tutorials and Coursebook readings. ☐ There are clear signs of independent research.	☐ Informal and formal proofs, as well as calculations, sometimes show intellectual initiative and insights beyond the content of tutorials and Coursebook readings. ☐ There are some signs of independent research.

Mark scheme for 2.2 (or Lower Second) and 3rd

Key Skill	2.2	3rd
Calculation Technique	☐ Algebra of sets: sets are sometimes constructed correctly – showing some understanding of the underlying operations and concepts. ☐ Logic: compound sentences are sometimes constructed correctly – showing some understanding of the syntax of the connectives. Truth tables are sometimes constructed correctly – showing some understanding of the relevant matrix construction techniques.	☐ Algebra of sets: sets are sometimes constructed correctly – showing some understanding of the underlying operations and concepts ☐ Logic: compound sentences are sometimes constructed correctly – showing some understanding of the syntax of the connectives. Truth tables are rarely constructed correctly – showing a poor understanding of the relevant matrix construction techniques.
Proof Technique	☐ Proof strategies are often present (implicitly or explicitly) and sometimes implemented correctly. Theorems and/or previous results, if invoked, are sometimes invoked and deployed correctly. Justification is given at most steps and is sometimes correct. Proofs are often presented in a tidy and accessible way.	☐ Proof strategies are sometimes present (implicitly or explicitly) and seldom implemented correctly. Theorems and/or previous results, if invoked, are seldom invoked and deployed correctly. Justification is given at some steps and is rarely correct. Proofs are sometimes presented in a tidy and accessible way.
Language	☐ Specialist vocabulary is often used and sometimes correctly.	☐ Specialist vocabulary is sometimes used, seldom correctly.
Creativity	☐ Proofs, as well as calculations, rarely show intellectual initiative and insights beyond the content of tutorials and Coursebook readings. ☐ There scarce signs of independent research.	☐ Informal and formal proofs, as well as calculations, rarely show intellectual initiative and insights beyond the content of tutorials and Coursebook readings. ☐ There are no signs of independent research.



1. Arguments

1.1 Introduction

Logic is the study of reasoning and seeks to establish what correct reasoning looks like. More specifically, logic is concerned with reasoning that can be expressed through language (if there are other kinds of reasoning, the logician is not interested in them).

Now, mostly, reasoning is carried out in *natural languages*: languages, that is, such as English, Italian, French, Japanese, that haven't been invented by anyone, but rather have spontaneously developed through time. On the other hand propositional logic, to which this handbook is an introduction, is a type of mathematical logic. As such, it does not deal directly with reasoning in natural languages. In order to use propositional logic, we first have to model it (represent it) in an appropriate *artificial* language – such as algebra, for example. Before we do that, however, we need to be familiar with what the model represents: reasoning in natural languages – in particular, in English.

This is why in this chapter you will see very little mathematics.

LEARNING OBJECTIVES

At the end of this chapter you will:

- know the definition of concepts such as *argument*, *validity* and *soundness*;
- be able to identify three types of arguments: valid and sound, valid and unsound, and invalid;
- know to what type of arguments proofs belong.

1.2 Arguments

Definition 1.2.1 — Argument. An argument is a finite sequence of sentences

$$s_1, s_2, \ldots, s_{n-1}, s_n$$

such that:

1. $s_1, \ldots, s_{n-1}$ are called *premises*;
2. s_n is called *conclusion*;
3. s_n is purported to follow from $s_1, \ldots, s_{n-1}$.

R Notice that an argument is not just any sequence of sentences. The following:

> My wife likes apples. I like apples, too. I also like candies.

is a sequence of sentences (and one that makes sense), but it is no argument. Can you say why, on the basis of Definition 1.2.1?

The notion of 'following' is meant to capture the idea that the premises of an argument back up its conclusion. However, it is not very precise (for the logician's standards). The standard way of making it precise is in terms of *logical consequence*.

Definition 1.2.2 — Logical consequence. A sentence q is a logical consequence of a sequence of sentences $p_1, \ldots, p_n$ if and only if there are no possible situations in which $p_1, \ldots, p_n$ are true and q is false.

Equipped with the notion of logical consequence, we can define:

Definition 1.2.3 — Validity. An argument is valid if and only if its conclusion s_n is a logical consequence of its premises $s_1, \ldots, s_{n-1}$.

■ **Example 1.1** Here is an example of a valid argument:

If it's summer, then the sun is out.
It's summer.
∴ The sun is out.

It is valid because, as per Definition 1.2.2, the conclusion – introduced by the symbol ∴, which means 'therefore' – is a logicl consequence of the premises. Which means, as per Definition 1.2.3, that there is no possible situation in which the premises are true and the conclusion false. And, indeed, there is no possible situation in which: 1) if it's summer then the sun is out, 2) it's summer, and 3) the sun is not out.

You might be having trouble accepting the first premise, *If it's summer then the sun is out* – as clearly it can be summer without the sun being out. But this is beside the point. When assessing the validity an argument, you have to *assume* the premises to be true and check whether, *on that assumption*, the conclusion might be false. ■

Furthermore:

Definition 1.2.4 — Soundness. An argument is sound if and only if it is valid and its premises are true.

■ **Example 1.2** Here is an example of a sound argument:

If I'm rich, then I don't have to work for a living.
I'm rich.
∴ I don't have to work for a living.

The previous example was of a valid but unsound argument, because one of its premises is false. This argument, on the contrary, is sound: because 1) there is no possible situation in which its premises are true and its conclusion false, and 2) its premises are true.

(Are the premises true? If you think there is room for dispute, have a go at designing an example of sound argument.) ■

The intuitive notion of an argument as a set of true premises that back up and guarantee the truth of a further sentence is, once made precise, that of sound argument.

R Notice that an argument is *invalid* if and only if its premises are true and its conclusion false.

R The previous remark entails that arguments with either false premises and true conclusion, or false premises and false conclusions, will always be valid. On the other hand, arguments with true premises and true conclusion are sound.

Exercise 1.1 Say whether any of the following two sequences of sentences is an argument.

The moon is made of cheese.
I'm rich.
$\therefore$ My sister is a blonde.

If you're English, then you're also British.
You're English.
$\therefore$ You like tea.

■

Exercise 1.2 Based on previous definitions, match the arguments on the left with the description on the right.

If the moon is made of cheese, then you can eat it. The moon is made of cheese. $\therefore$ You can eat it.	INVALID
If $4 = 2+2$ and $2 = 1+1$, then $4 = 1+1+1+1$. $4 = 2+2$ and $2 = 1+1$. $\therefore$ My sister is a blonde	SOUND
If you're English, then you're also British. Queen Elizabeth II is English. $\therefore$ Queen Elizabeth II is British.	VALID AND UNSOUND

■

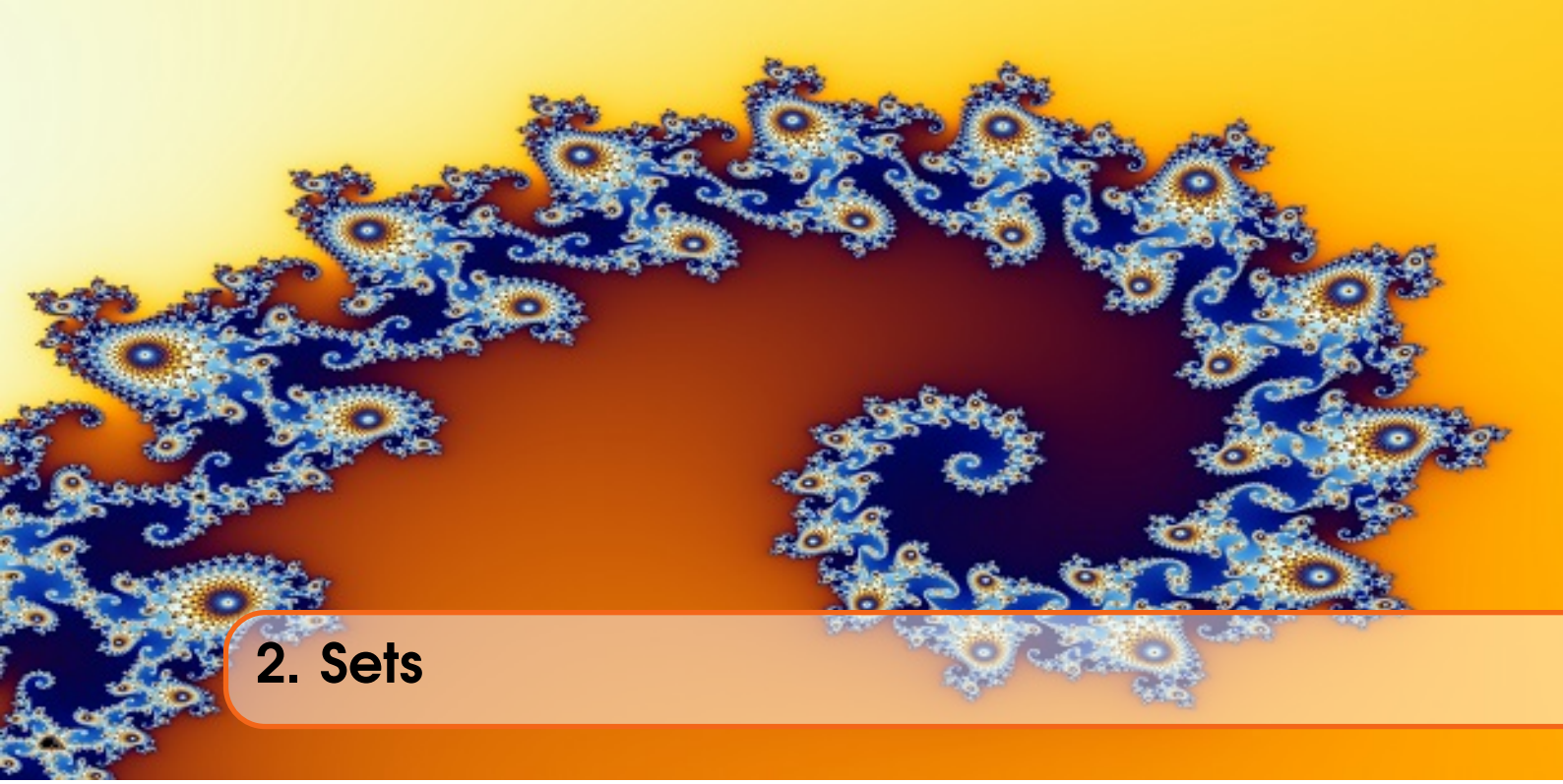
1.3 Homework

[This is where you should attach your homework once it has been marked.]

[This is where you should attach your homework once it has been marked.]

[This is where you should attach your homework once it has been marked.]

1.4 Notes



2. Sets

2.1 Introduction

You will remember from you Maths lessons what a set is: things bunched together. This is how Georg Cantor (1845-1918), a German mathematician and the founder of set theory, defined the concept of set:

> By a 'set' we are to understand any collection into a whole M of definite and separate objects of our perception or our thought. These objects are called the 'elements' of M.

Nowadays mathematicians work with other definitions. This, however, will do for our purposes. You have a set, then, when you group together some given objects and consider the group as a whole. So in a sense a set is a new object: there are *many* pens in this room, for example, but there is only *one* set of all the pens in this room.

In this chapter we will look at some basic set theory – actually, the part of set theory called *algebra of sets* – and then learn how to prove stuff about sets.

LEARNING OBJECTIVES

At the end of this chapter you will:

- be familiar with the notation of the algebra of sets;
- be able to construct sets via the three operations of intersection, union and complement;
- have seen the laws of the algebra of sets
- be able to read and understand proofs about sets;
- be able to work out simple proofs about sets.

2.2 Some Basics

Since – recall Cantor's definition – a set is a collection of things, it makes sense that the fundamental notion in set theory should be *set membership*: the relation between an object which is a member, or an element, of a set, and the set itself. For example, my cat Olivia is a member of the set of all the cats in Europe, and Hertswood Academy is a member of the set of all the schools in England.

You can also say that Olivia and Hertswood are 'in' those sets or that they 'belong' to those sets. Avoid the word 'contained': that's a different set-theoretical relation.

To express the fact that an object x belongs to a set A, we write $x \in A$. To express the fact that x does not belong to A, we write $x \notin A$. So:

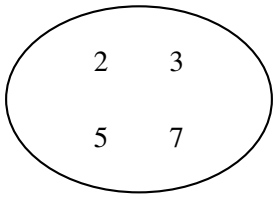
Definition 2.2.1 — Set membership. If x is an object and A a set,
$x \in A$ means 'x is a member of A',
$x \notin A$ means 'x is not a member of A'.

Importantly, two sets are identical, written $A = B$, if they have the same members:

Definition 2.2.2 — Identity between sets. For any two sets A, B, $A = B$ if and only if:
for every $x \in A$, $x \in B$;
for every $x \in B$, $x \in A$.

There are at least three ways to describe – or, as I will also say, "give" – a set: with Venn diagrams, by enumeration, and by characteristic.

You should already by familiar with the first way. It is the more visual of the three. Venn diagrams are ellipses (egg-shaped closed curves) representing the collecting that gives rise to the set. All members of the set are inside it, while objects that are not members of the set are outside. The Venn diagram of the set P of the first four prime numbers, for example, looks as follows:



For our purposes, Venn diagrams may be useful to visualise set-theoretic laws. However, are pretty useless when it comes to either big sets or sets with a complex construction (we will see what that means). Also, they are little use in proofs.

The next way to give a set is by enumerating (listing) its members. This is done by the brace notation. If we want to give the set P of the four prime numbers, for example, we are going to write:

$$P = \{2, 3, 5, 7\}$$

Clearly, listing the members of a set will only take you so far. For example, you can't give the set N of the natural numbers by enumeration, because that set has infinitely many members. Also, in some cases (e.g. in proofs) you won't be interested in the particular members of a set, but only in the feature that makes them belong to the set. For example, we might want to say that the set of the first four prime numbers is smaller, in some sense of 'smaller', than the set of the first twenty prime numbers; in this case we won't be interested in what the first prime numbers are in particular.

This brings us to the third and best way to give a set, and the one we will mostly be using. What we do is to give a set by specifying the characteristic, or the feature, or the property, or the condition that the members of the set have, or meet, and that makes them members of the set. In our present case, for example, that condition will be *being one of the four prime numbers*. So:

$$P = \{x \mid x \texttt{ is one of the first four prime numbers}\}$$

This reads: 'P is the set of the x such that x is one of the first four prime numbers. In general, if ϕ is a sentence expressing any condition on x, a set P can be given as follows:

$$P = \{x \mid \phi\}$$

Instead of |, a colon may be used. So:

$$P = \{x\text{: is one of the first four prime numbers}\}$$
$$P = \{x : \phi\}$$

For example, if P is the set of the natural numbers between 3 and 13, we will set ϕ as '$x \in \mathsf{N}$ and $3 < x < 13$' and write:

$$P = \{x \mid x \in \mathsf{N} \text{ and } 3 < x < 13\}$$

where N is the set of all the natural numbers.

Exercise 2.1 Give the following sets:

- the set of the red houses in England
- the set of the even numbers between 11 and 32
- the set of the squares with an area larger than 25 m^2

as Venn diagrams, as lists, and by condition. ■

Two important sets are the empty set $\emptyset$ and the universal set U. Intuitively, the empty set is the set with nothing in it, while the universal set is the set with everything in it (it is our 'domain of discourse'). They can both be given by condition. The empty set is the set of the x such that $x \neq x$, while the universal set is the set of the x such that $x = x$ (why do these condition give exactly those sets?)

Definition 2.2.3 — Empty set.

$$\emptyset = \{x \mid x \neq x\}$$

R Notice that $\emptyset \neq \{\emptyset\}$: the former is the set with no members, the latter is a set with one member (that member being the empty set). Moreover, there is a set $\{\emptyset, \{\emptyset\}\}$: its members are the empty set and the set whose sole member is the empty set. And we could go on forever. Even though it is rather irrelevant to our purposes, this is one of the most important facts in mathematics, and it lies at the very foundations of the discipline. To find out why: `https://www.youtube.com/watch?v=6UWhPnbZv-o`

Definition 2.2.4 — Universal set.

$$\mathsf{U} = \{x \mid x = x\}$$

R I said that, intuitively, the universal set is the set with everything in it. In fact, this is true only in a qualified sense. The universal set is the set with everything *we are allowed to talk about* in it. So while the universal set can be either very small or very, very big – depending on what we are allowed to talk about – in any case it has to be limited (we are not allowed to talk about everything). For our purposes, this is scarcely relevant. You should, however, be aware of it.

You might want to know why the universal set has to be limited. Why on earth can't we talk about *absolutely everything*? Briefly, the reason is that if you allow the existence of truly universal, all-comprehensive set, you run into trouble. I will leave it to you to find out more, if you are interested. There is a short but decent Wikipedia entry on this: `https://en.wikipedia.org/wiki/Universal_set`. You might struggle to read it, because the notion of *subset*, which I haven't or will introduce here, is central to the problem. Have a go at researching that, too. The reason why I am asking you to research this independently is that you will be required to do so at university. Besides, you can always come to me, or email me, if you need help.

Now we come to the three main operations of the algebra of sets: intersection, union and complement. You should already be familiar with them too.

The intersection of two sets A and B, written $A \cap B$ is the set whose members are the members of *both* A and B (those, that is, that A and B have in common). So:

Definition 2.2.5 — Intersection.

$$A \cap B = \{x \mid x \in A \ \texttt{ and } \ x \in B\}$$

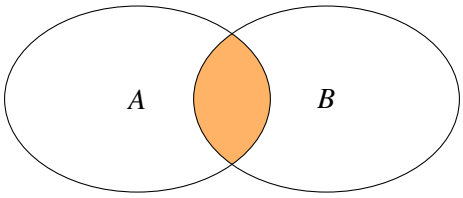
If $A \cap B = \emptyset$, then A and B are *disjoint*:

Definition 2.2.6 — Disjoint sets.

$$A \upharpoonright B := A \cap B = \emptyset$$

where the symbol $:=$ reads 'is defined as'.

Here is a Venn representation of $A \cap B$ (shaded area) for any sets A and B:



■ **Example 2.1** Let $A = \{1,2,3,4\}$ and $B = \{3,4,5\}$. Then $A \cap B = \{3,4\}$. ■

■ **Example 2.2** Let $A = \{1,2,3,4\}$ and $B = \{5\}$. Then $A \cap B = \emptyset$. ■

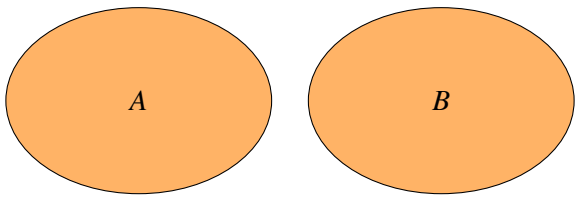
Exercise 2.2 Choose two sets and give their intersection as a Venn diagram, by enumeration, and by condition. ■

The *union* of two sets A and B, written $A \cup B$, is the set whose members are the members of *either* A *or* B: this means that in $A \cup B$ you will find everything that is in A and everything that is in B. So:

Definition 2.2.7 — Union.

$$A \cup B = \{x \mid x \in A \ \texttt{ or } \ x \in B\}$$

Here is a Venn representation of $A \cup B$ (shaded area) for any sets A and B:



Here A and B are represented as disjoint. They need not be.

■ **Example 2.3** Let $A = \{1,2,3,4\}$ and $B = \{27,31\}$. Then $A \cup B = \{1,2,3,4,27,31\}$. ■

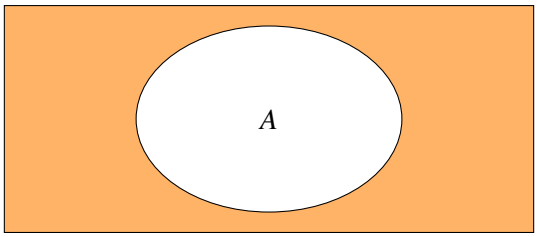
Exercise 2.3 Choose two sets and give their union as a Venn diagram, by enumeration, and by condition. ■

Finally, the *complement* of a set A, written A' (sometimes $\overline{A}$), is the set whose members are not in A, i.e., the set with everything that is not in A in it. So:

Definition 2.2.8 — Complement.

$$A' = \{x \mid x \notin A\}$$

Here is a Venn representation of A' (shaded area) for any set A:



The rectangle represents U, the universal set. As expected, the A' is U minus A.

■ **Example 2.4** Let $A = \{1,2,3,4\}$ and $\mathsf{U} = \{x \mid x \in \mathsf{N} \text{ and } 1 \leq x \leq 5\}$. Then $A' = \{5\}$. ■

Exercise 2.4 Choose a set and give its complement as a Venn diagram, by enumeration, and by condition. ■

2.3 Constructing Sets

It is important to understand two things. First, that the intersection and the union of two sets, as well as the complement of a set, are *themselves* sets. This means that all three operations can be applied to them – and again to the result of *this* application, and so on. So if A, B, C are any three sets, the following sets always exist:

$$(A \cap B) \cap C$$
$$(A \cap B) \cup C$$
$$(A \cap B)'$$

and so does, for example, the following:

$$((A \cap B) \cap C)' \cup B'$$

The second important thing is that the three operations can be applied to *any set whatsoever*. Another way of saying this (recall your algebra) is that A and B, in Definitions 2.2.4, 2.2.5 and 2.2.6, are variables that can take any set as their value. Even the empty and the universal set. This is thus a genuine set:

$$\emptyset \cap \mathsf{U}$$

It is the intersection between the empty set and the universal set. The following are also perfectly acceptable sets:

$$\emptyset'$$
$$(\mathsf{U} \cup A) \cap \emptyset$$

The latter is, in English, the intersection between the union of the universal set and A, and the empty set.

Exercise 2.5 Construct three sets by applying the three operations $\cap, \cup, '$ three times to each of the following sets (so you will end up with 3 new sets, but will have constructed 9 in total: 3 final plus 6 transitional ones).

- A'
- $\emptyset \cup A$
- $(B \cap C) \cup \mathsf{U}$

■

Let's see a few concrete examples. Let $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7\}$, $C = \{2,3,8,9\}$. Then:

$$A \cup B = \{1,2,3,4,5,6,7\}$$
$$A \cup C = \{1,2,3,4,8,9\}$$
$$B \cup C = \{2,3,4,5,6,7,8,9\}$$
$$A \cap B = \{3,4\}$$
$$A \cap C = \{2,3\}$$
$$B \cap C = \{3\}$$

Exercise 2.6 Let $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7\}$, $C = \{5,6,7,8,9\}$, $D = \{1,3,5,7,9\}$, $E = \{2,4,6,8\}$, $F = \{1,5,9\}$. Work out:

- $A \cup B$ and $A \cap B$
- $A \cup C$ and $A \cap C$
- $D \cup F$ and $D \cap F$

■

2.4 The Laws of the Algebra of Sets

You may have wondered, upon seeing $\emptyset \cap \mathsf{U}$, what that set looks like. It is not like have two ordinary bunches of things which may or may not have some members in common: $\emptyset$ is empty, and U is... well, full. What's their intersection?

The laws of the algebra of sets answer this and other questions. What they do is to equate notable (i.e., particularly remarkable) sets constructed through the three operations to other, identical sets (recall that two sets are identical if they have the same members). This is useful because, as we have just seen, it is not always obvious what a set such as $\emptyset \cap \mathsf{U}$ actually looks like.

There are quite a few of these laws. For the time being we will just state them and learn how to read them. In the next section we will see how we can prove them from the definitions of the three operations. Bear in mind that A, B, C are arbitrary sets.

If they look scary, don't worry; in fact, they're quite tame. Here we go:

Laws of the Algebra of sets — Associative laws

S1.1 $(A \cap B) \cap C = A \cap (B \cap C)$

S1.2 $(A \cup B) \cup C = A \cup (B \cup C)$

Laws of the Algebra of sets — Commutative laws

S2.1 $A \cap B = B \cap A$

S2.2 $A \cup B = B \cup A$

Laws of the Algebra of sets — Absorption Laws

S3.1 $A \cap (A \cup B) = A$

S3.2 $A \cup (A \cap B) = A$

Laws of the Algebra of sets — Distributive Laws

S4.1 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

S4.2 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Laws of the Algebra of sets — Neutral Elements Laws

S5.1 $A \cap \mathsf{U} = A$ **S5.2** $A \cap \emptyset = \emptyset$

S5.3 $A \cup \mathsf{U} = \mathsf{U}$ **S5.4** $A \cup \emptyset = A$

Laws of the Algebra of sets — Complement Laws

S6.1 $A \cap A' = \emptyset$

S6.2 $A \cup A' = \mathsf{U}$

Here is how you should approach them. Choose one – say S1.1, the first of the Associative Laws. It states that the intersection between, on the one hand, the intersection of A and B, and on the other hand C, is the same set as the intersection of, on the one hand, A, and on the other hand the intersection between B and C.

If you are not convinced, that's great: we will prove the law in the next section. Trust me for the time being. What is important for now is that you are able to read and understand the claim that each law makes. Let's look at a couple more.

Take S6.1, the first Complement Law. It states that the intersection between A and its complement A' is the empty set (i.e., is empty). Sounds more like it?

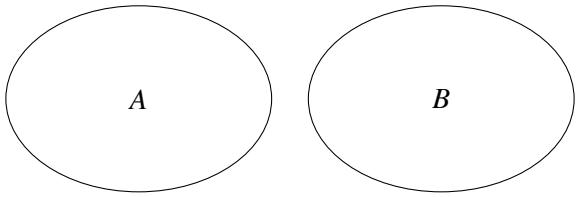
Finally, S3.2, the second Absorption Law. It states that the intersection between, on the one hand, A, and on the other hand the union of A and another set B, is the same set as A. Confused again? Here is an exercise to help you familiarise with all this.

Exercise 2.7 Choose two Laws of the Algebra of Sets (except S1.1, S6.1 and S3.2) and translate them into English. ■

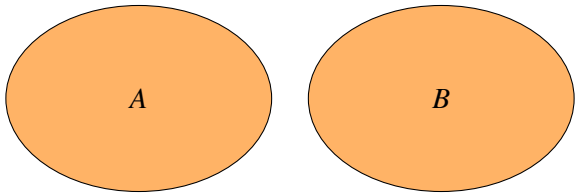
2.5 Venn Diagrams and Insight into the Laws

It takes some work even to understand what S1.1–S6.2 mean, let alone being convinced that they are true. Venn diagrams can provide more insight. Take for example the first Absorption Law, S3.1:

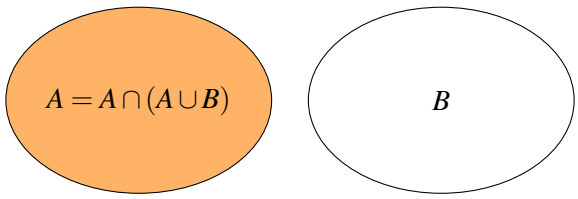
$A \cap (A \cup B) = A$. If the Venn diagram of the left-hand side turns out to be identical with that of the right-hand side, then S3.1 must be true. Let's put A and B in place:



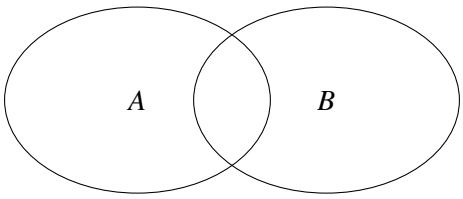
Start with $A \cup B$:



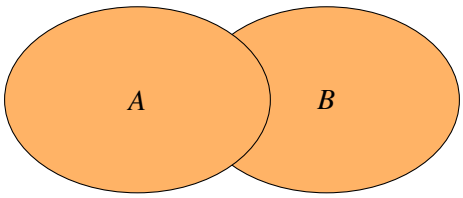
If we now intersect the orange area with A, what we are left with is simply A (because it is what $A \cup B$ and A have in common):



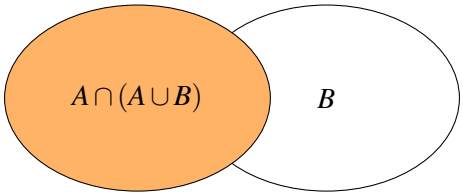
At the outset, we set A and B as disjoint. You might think it has something to do with our result. It doesn't. Let's set them as intersecting (i.e., let's set $A \cap B \neq \emptyset$):



Then $A \cup B$ will look like this:



If we now intersect it with A:



we get A again.

Exercise 2.8 Take the laws you chose for Exercise 2.7 and use Venn diagrams to show that they hold. ■

2.6 How to prove Stuff about Sets

While giving more insight into S3.1, however, this is hardly a proof for it: as we know, a proof for S3.1 is a valid and sound argument having S3.1 as conclusion. We will now see how to work out one.

Recall that two sets, no matter how complex their construction, are identical if they have the same memebers. Thus, to prove that a set as complex as $A \cap (A \cup B)$ is identical with A, it is enough that we show that an arbitrary object x belongs to one if and only if it belongs to the other.

To do this, typically there are three main steps:

- translate everything in terms of the definition of the relevant complex sets;
- assume that an arbitrary object x is in the left-hand side set, and show that it is therefore also in the right-hand side set;
- assume that an arbitrary object x is in the right-hand side set, and show that it is therefore also in the left-hand side set.

A proof thus structured is called a *direct proof*. Here is an example:

Proof S3.1: $\mathbf{A} \cap (\mathbf{A} \cup \mathbf{B}) = \mathbf{A}$

Prove that $A \cap (A \cup B) = A$.

Proof:

Left to right:

1. Assume that x belongs to $A \cap (A \cup B)$ (Assumption);
2. then x belongs to both A and $A \cup B$ (by step 1 and Definition 2.2.4);
3. if x belongs to $A \cup B$, then either x belongs to A or x belongs to B (by Def. 2.2.6);
4. then x belongs to both A and either A or B (by steps 2 and 3);
5. therefore, x belongs to A (by step 4). □

Right to left:

6. Assume that x belongs to A (assumption);
7. then x belongs to $A \cup B$ (by Def. 2.2.6);
6. therefore, x belongs to $A \cap (A \cup B)$ (by Def. 2.2.4). ■

Some remarks. First, □ and ■: they tell readers, respectively, when a stage of the proof is over, and when the whole proof is ended. Second, at every step a justification is invoked; at step 1, for example, Definition 2.2.6. At step 3, previous steps 1 and 2. Besides definitions and previous steps, axioms (laws assumed to be true) and available theorems (proven laws) may be invoked as justifications. Your job as a reader is to find the definition, the step, the axiom or the theorem and check that it does indeed justify the step. This may not be immediate; please keep at it until you have your little 'Eureka!' or light-bulb moment. Your job while you work out the proof is to always invoke a justification and to make sure that it does indeed justify the step at which you invoke it.

Finally, as you can see, getting for the more to the less complex is somewhat more difficult, because you have to think backwards. Getting from the less to the more complex is comparatively easy: you only have to make sure you construct sets according to the rules.

Let us look at another example, L6.1 (the first Absorption Law). The procedure here is slightly different, because the empty set is involved. To get started, however, it will be helpful to prove that there is only one empty set, i.e., that any set with no members is identical with $\emptyset$. I will start using set-theoretic notation a little more.

Theorem 2.6.1 — Unicity of $\emptyset$. For every set A, if $x \notin A$ for every object x, then $A = \emptyset$.

Proof Theorem 2.6.1: For every set A, if $x \notin A$ for every object x, then $A = \emptyset$

Proof:

1. Assume that, for an arbitrary set A and for any object x, $x \notin A$ (assumption);
2. the assumption, then, is that A has no members (clarification);
3. then $A = \emptyset$ (by step 1, Defs. 2.2.2 and 2.2.3). ■

Now we can prove L6.1:

Proof S6.1: $\mathbf{A \cap A' = \emptyset}$

Prove that $A \cap A' = \emptyset$.

Proof:

1. Assume that $x \in A$, for x arbitrary (assumption);
2. then $x \notin A'$ (by step 1 and Def. 2.2.8);
3. assume that $x \in A'$ (assumption);
4. then $x \notin A$ (by step 3 and Def. 2.2.8)
5. then $x \notin A \cap A'$ (by step 2 and Def. 2.2.5);
6. notice that since x is any object whatsoever, step 3 means that $A \cap A'$ has no members (clarification);
7. therefore, $A \cap A' = \emptyset$ (by step 3 and Theorem 2.6.1). ■

Notice that clarifications are not justifications (step 6, for example, is not invoked at step 7); however, they help struggling readers. Also notice that in this case it was unnecessary to distinguish between two directions: we exhausted all cases in steps 1–4 and then used Theorem 2.6.1 to get '='. It is still, however, a case of direct proof.

A different, very powerful proof technique is *proof by contradiction* (also known as *reductio ad absurdum* or *RAA*). You want to prove a certain claim x. To do so, you assume the negation of x, derive a contradiction, and conclude 1) that the negation of x is false (because it entails a contradiction), and 2) that since the negation of x is false, then x is true.

Let's prove $A \cap A' = \emptyset$ by contradiction:

Proof S6.1: $\mathbf{A \cap A' = \emptyset}$

Prove that $A \cap A' = \emptyset$.

Proof:

1. Assume $A \cap A' \neq \emptyset$. (Assumption)
2. Then there is at least an x such that $x \in A \cap A'$. (Step 1, Def. 2.2.3)
3. If $x \in A \cap A'$, then $x \in A$ and $x \in A'$. (Def. 2.2.5)
4. If $x \in A \cap A'$, then x belongs to *both* A and A'. (Clarification)
5. If $x \in A$, then $x \notin A'$. (Def. 2.2.8)
6. If $x \in A'$, then $x \notin A$. (Def. 2.2.8)
7. There is no x such that $x \in A$ and $x \in A'$. (Steps 5, 6)

8. There is no x such that $x \in A \cap A'$. (Steps 3, 7)
9. Contradiction. (Steps 8, 2)
10. Therefore, $A \cap A' = \emptyset$. ■

R One thing this tells you is that there are often more than one way to prove a given claim – even in mathematics.

Exercise 2.9 Choose 5 Laws (including the two you have already worked on in previous exercises, but excluding S3.1 and S6.2) and have a go at proving them. ■

2.7 Homework

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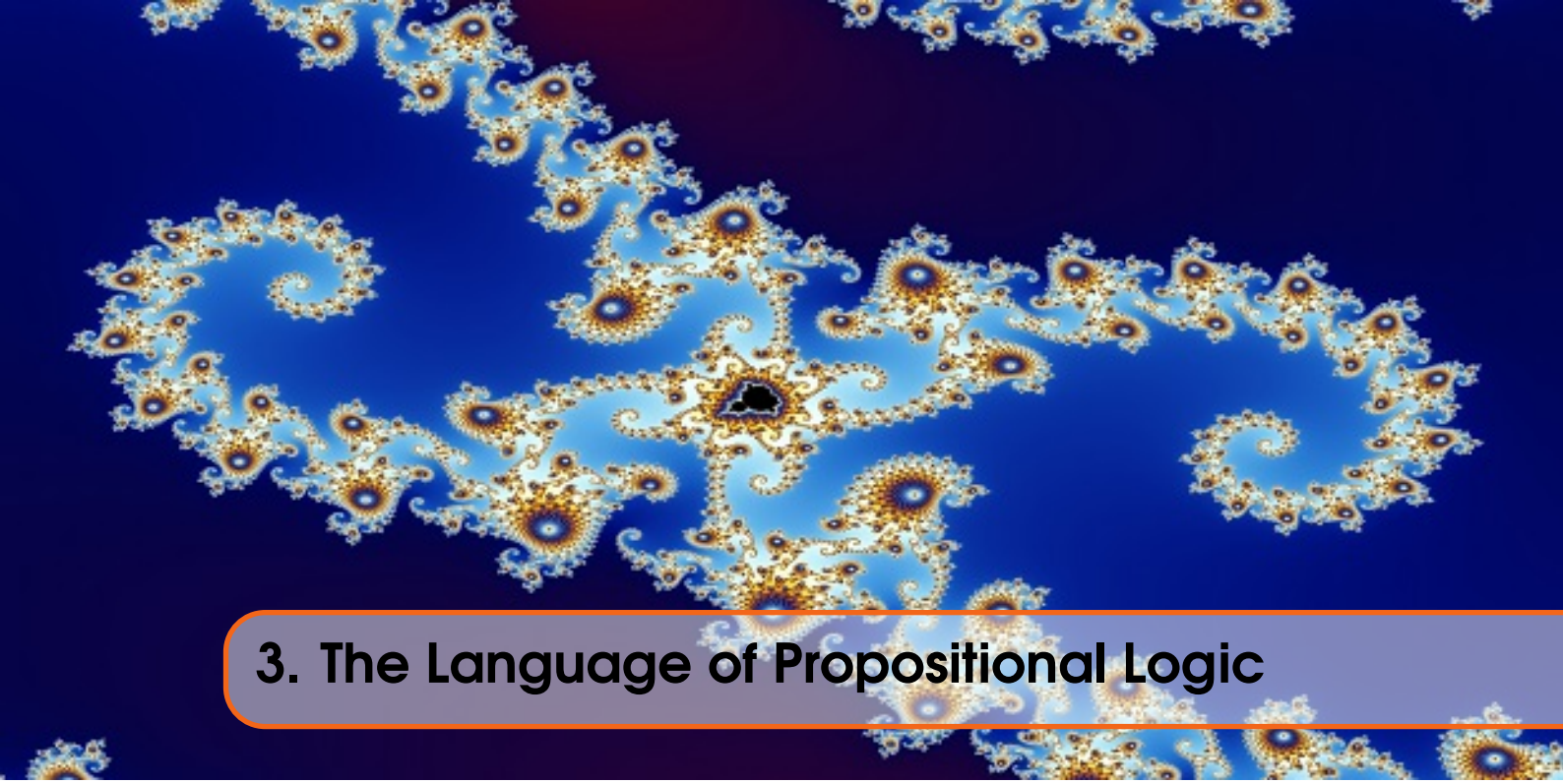
2.8 Notes



3. The Language of Propositional Logic

3.1 Introduction

Propositional logic is a mathematical representation – or *model* – of reasoning in natural languages such as English, Spanish, Italian, Swedish, Japanese, and so on. The modelling process has two stages: the modelling of language, and the modelling of reasoning. The parts of propositional logic in which language is modelled are *syntax* and *semantics* ; the part in which reasoning is modelled is called *proof theory* . Syntax has to do with how the building blocks of language, namely, sentences, must be built. Semantics has to do with meaning (what sentences mean). Proof theory has to do with how to correctly put sentences together in order to construct valid arguments. Propositional logic has a number of proof theories, spanning from very minimal to quite complex, and each of them gives rise to a number of so-called 'formal deductive systems'.

We will only look at syntax and semantics. We will, however, still be able to prove things (i.e., construct valid and sound arguments). The only difference is that we will do so mathematically, or informally, rather than proof-theoretically, or formally. To give you a feel of the distinction, here is what a formal-deductive proof looks like (in a system called Fitch):

1. A	Ass.
2. B	Ass.
3. A	Reit 1
4. B $\rightarrow$ A	CondIntro: 2-3

As you can see, there is no English discourse: just numbers on the left (steps), formulas in the middle (sentences) and application of rules on the right. This makes formal deductive proofs extremely rigorous. A mathematical proof, on the other hand, looks like those we have seen in Chapter 2, with both symbols and English in them. Now, every correct mathematical proof can be reconstructed as a formal deductive proof; in fact, the whole of mathematics can be so reconstructed. Typically, however, mathematicians are not interested in doing so – though they're generally happy to know that they could if they wanted to. We will follow them here.

LEARNING OBJECTIVES
At the end of this chapter you will:
- be familiar with the language of propositional logic;
- know the difference between atomic and compound sentences and be able to construct sentences of increasing complexity through the use of the connectives;
- be familiar with the concept of truth-functionality and able to understand the truth tables of $\wedge$, $\vee$, $\neg$.

3.2 Kinds of Sentences, Connectives and Truth-Functionality

Just as in algebra we work with variables in place of numbers, and in the algebra of sets we work with variables in place of specific sets (recall that to prove, say, L5.2 we need not know what set A is), so in propositional logic we work with *sentence letters* such as p, q, r, s, t rather than specific sentences. These are variables just as you usual algebraic a, b, c or x, y, z.

Sentence letters, however, typically stand for *atomic sentences*. These are sentences that have no part that is itself a sentence. E.g., *It is raining* is a atomic, while *It is raining and the wind is blowing* is not atomic, because it has two parts, *It is raining* and *The wind is blowing*, that are themselves sentences.

Definition 3.2.1 — Atomic sentence. A sentence is *atomic* if none of its parts is a sentence.

Even though atomic sentences have no sentences as their parts, they still do have parts: nouns, verbs, etc. In propositional logic, however, we disregard these; in other words, we never go more fundamental than sentences. This is another sense in which atomic sentences are 'atomic': in propositional logic, they are the atoms of language. This is, if you like, a limit of the model; it is also, however, one of its strengths.

Definition 3.2.2 — Complex sentence. A sentence is *compound* if it is not atomic.

Compound sentences result from the application of certain operators connecting atomic sentences. This happens in English, too. Our earlier example, *It is raining and the wind is blowing*, is a case in point: it results from the application of the two-place conjunction *and* to the atomic sentences *It is raining* and *The wind is blowing*. *It is raining or the wind is blowing* would also be a good example. Importantly, in propositional logic operators such as *and* and *or only connect sentences* – which of course is unsurprising, given that, in propositional logic, sentences is all we have. In English this is not true: I can say, for example, *I like ice cream and chocolate*, where *and* connects nouns.

Definition 3.2.3 — Logical connective. A *logical connective* is an operator that takes sentences to form new and more complex sentences.

In propositional logic, the logical connectives are *truth-functional*. This means that the truth value of compound sentences formed through the connectives is completely determined by the truth value of their components.

Definition 3.2.4 — Truth value. Every sentence can be associated with exactly one out of two truth values: True or False.

Definition 3.2.5 — Truth-functionality. A logical connective is *truth-functional* if the truth value of compound sentences formed through it is completely determined by the truth value of their components.

The connectives may thus be seen in two ways. One way is as glue that holds sentences together. Another way is as an input-output machine that takes in truth values and put out truth values

according to some rule. In propositional logic, the meaning of the connectives is specified in terms of this rule.

3.3 Connectives and Truth tables

We are interested in three logical connectives: conjunction, written $\wedge$; disjunction, written $\vee$; negation, written $\neg$. Before we look at them, however, we need to familiarise with truth tables.

Truth tables are two-way tables constructed as follows. Suppose you want to check the truth value of a compound sentence formed out of two atomic sentences connected by the 2-place operator ♠ (not a real operator, clearly). Then you would construct a two-way table listing the three sentences – two atomic and one compound – horizontally and all the possible combinations of truth values vertically. You would start like this: and then fill in the third column according to how ♠

p	q	p ♠ q
T	T	?
T	F	?
F	T	?
F	F	?

transfers truth values to a compound sentences from its components. Importantly, each connective is *entirely and uniquely determined* by its truth table. ♠ is a fictional connective; let's have a look at a real one.

Connectives — Conjunction, $\wedge$

Conjunction, or $\wedge$, is the two-place, truth-functional sentential connective with the following truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

R According to the definition, $\wedge$ takes two sentences to make a more complex sentence. The new sentence will be true if and only if *both* the original sentences are true.

Connectives — Disjunction, $\vee$

Disjunction, or $\vee$, is the two-place, truth-functional sentential connective with the following truth table:

p	q	$p \wedge q$
T	T	T
T	F	T
F	T	T
F	F	F

R According to the definition, $\vee$ takes two sentences to make a more complex sentence. The new sentence will be true if and only if at least one of the original sentences is true. This, notice, means that $\vee$ is an *inclusive* disjunction: according to the connective's truth table, for $p \vee q$ to be true p and q may be both true.

Connectives — Negation, $\neg$

Negation, or $\neg$, is the one-place, truth-functional sentential connective with the following truth table:

p	$\neg p$
T	F
F	T

R According to the definition, $\neg$ takes one sentence to form a new, more complex sentence. The new sentence is true if and only if the original sentence is false. Notice that, unlike $\wedge$ and $\vee$, $\neg$ is a *one-place* connective. This stretches the notion of a 'connective' a little, but this is common procedure in mathematics.

We will see more about truth tables in the next chapter (in fact, we will use them extensively). For the remainder of this chapter, we will be concerned with constructing new sentences by using the connectives.

It is always possible to construct new sentences by using $\wedge, \vee, \neg$. It works just as constructing sets with $\cap, \cup, '$. And just as with sets, one must be careful and write things properly. For example, just as there is no set $A \cup \cap B$, there is no sentence $p \vee \wedge q$. There are, however, a set $A \cap (A \cup B)$ and a sentence $p \wedge (p \vee q)$.

A few rules to keep in mind:

- Always make sure that a two-place connective is actually connecting two sentences – so that $p \wedge q$ is well-formed, while $p \wedge$ is not (it would be like writing $2+$ as opposed to, say, $2+2=4$).
- When connecting compound sentences (with compound or atomic sentences), you must make sure they are within parentheses – as in $p \wedge (p \vee q)$. This is because otherwise sentences are ambiguous: how should $p \wedge p \vee q$ be parsed?
- Even though, as per the previous rule, strictly speaking one should write $p \wedge (\neg q)$, because $\neg q$ is a compound sentence, it is permissible and in fact customary to write $p \wedge \neg q$. (Why might that be?).

■ **Example 3.1** From atomic sentence p we may construct:

$\neg p$
$\neg\neg p$
$(p \wedge p) \vee p$

From atomic sentences p, q:

$p \wedge q$
$\neg p \wedge q$
$\neg(p \wedge q)$ (Do you understand why this is different from the previous one?)
$(p \vee q) \wedge (p \vee \neg q)$

Each of these sentences will have its own truth table – but you need not worry about that now. ■

Exercise 3.1 Construct at least three new sentences from the following ones:

p
$p \wedge q$
$\neg(q \wedge \neg q)$

Feel free to use any additional atomic sentences as you feel appropriate. ■

3.4 Homework

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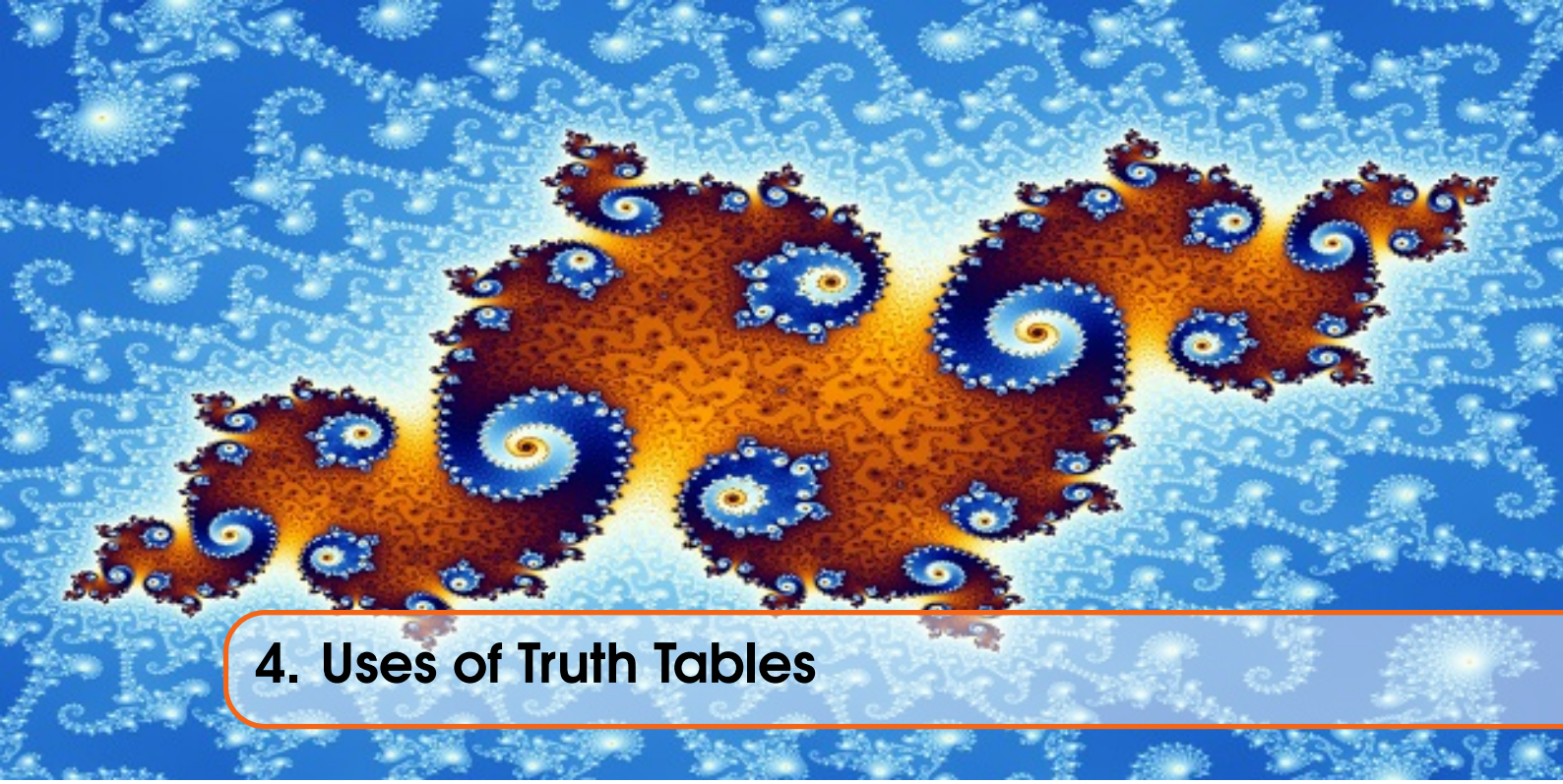
3.5 Notes



4. Uses of Truth Tables

4.1 Introduction

Propositional logic deals with propositional variables (sentence letters) rather than with specific propositions. It follows that the logician is interested in mathematically studying the validity, rather than the soundness, of arguments. (To see why, recall Definitions 1.2.3 and 1.2.4.) One way to study the validity of a given argument is with truth tables. In this chapter we will mainly look at how to do so. Truth tables have also another use – that of checking tautologies – which we will also look at.

LEARNING OBJECTIVES

At the end of this chapter you will:

- be able to construct truth tables for compound sentences;
- be able to use truth tables to check the validity of arguments;
- be familiar with the notion of tautology and be able to use truth tables to identify tautologies.

4.2 Constructing truth tables

In order to construct truth tables for compound sentences we need to be familiar with the notion of main connective. To do so, let us define:

Definition 4.2.1 — Scope of a connective. The scope of a logical connective is the sentences that it connects, be these atomic or compound.

■ **Example 4.1** The scope of connective $\wedge$ in $p \wedge q$ is constituted by sentences p and q, and is the whole compound sentence. So:

$$\underline{p \wedge q}$$

However, the scope of $\vee$ in $p \wedge (q \vee r)$ is not the whole compound sentence, but only the parenthesis:

$$p \wedge \underline{(q \vee r)}$$

The scope of $\wedge$ is still the whole compound sentence:

$$\underline{p \wedge (q \vee r)}$$

Thus, in the second sentence $\wedge$ has a wider scope than $\vee$. In the following sentence, however, $\neg$ has a wider scope than both $\wedge$ and $\vee$:

$$\underline{\neg(p \wedge (q \vee r))}$$

■

Exercise 4.1 Identify the scope of each connective in each of the following sentences.

$$p \vee (\neg q \wedge p)$$
$$\neg q \vee \neg(p \vee (q \wedge \neg r))$$
$$\neg q$$

■

Definition 4.2.2 — Main connective. The main connective in a given compound sentence is the connective with the wider scope.

■ **Example 4.2** As we have seen, in $p \wedge (q \vee r)$ the connective with the wider scope is $\wedge$. Thus, it is the main connective. ■

Exercise 4.2 Identify the main connective in each of the following sentences.

$$p \vee (\neg q \wedge p)$$
$$\neg q \vee \neg(p \vee (q \wedge \neg r))$$
$$\neg q$$

■

Now we are ready to see how to construct truth tables for ever more complex sentences. Take our usual $p \wedge (q \vee r)$, for example. First of all, we identify the atomic sentences that go into it. These are p, q, r. Then we put connectives in order of (increasing) scope: $\vee$, $\wedge$. We can thus see the target sentence $p \wedge (q \vee r)$ as made up of different layers: first the atomic sentences p, q, r; then the compound sentence governed by the connective with the narrowest scope, namely, $q \vee r$; finally, the compound sentence governed by the connective with the widest scope, namely, the whole target sentence $p \wedge (q \vee r)$.

So now we have five sentences in total, ordered by complexity:

$$p, q, r$$
$$q \vee r$$
$$p \wedge (q \vee r)$$

Our truth table will thus have five columns (the vertical line separates atomic sentences from compound ones):

p	q	r	$q \vee r$	$p \wedge (q \vee r)$

How many rows will it have? As many as the combinations of truth values available for the three *atomic* sentences. So:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Now all we have to do is to work out, based on our definitions of the logical connectives in Chapter 3, the truth value for each of the next sentences in the table. For example, in the first row we will look at the truth values of q and r, which are T and T respectively, and ask ourselves, 'What is the truth value of $q \vee r$ when q and r are both true?'. The answer, again, comes from the definition of $\vee$, and is: T. So:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	
T	T	F	T	
T	F	T	T	
T	F	F	F	
F	T	T	T	
F	T	F	T	
F	F	T	T	
F	F	F	F	

And finally (and this time we have to look at the truth values for $q \vee r$ and p, as well as the definition of $\wedge$):

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Exercise 4.3 Construct the truth tables of the following sentences.

$\neg(p \wedge q)$
$\neg p \wedge \neg q$
$p \vee (p \wedge q)$

■

4.3 Checking validity with Truth Tables

To check whether an argument is valid, we need to check whether it is possible for its premises to be true and its conclusion false. We can do so easily by means of truth tables.

Suppose we have the following argument:

p
$p \wedge q$
$\therefore \neg(p \wedge q)$

We construct a truth table as we have learnt in the previous section:

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Let us now highlight the premises of the argument (ocre) and its conclusion (blue):

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

We are interested in the rows where both premises are true. This is only the first row:

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Is the conclusion true or false in that row? It is false:

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Thus, not only is it possible for the premises to be true and the conclusion false: it is the only possibility. Therefore, the argument is very much invalid.

Here is another example. Consider the following argument:

p
$\neg p \vee q$
$\therefore q$

Its truth table is as follows:

p	q	$\neg p$	$\neg p \vee q$	q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

I have re-written q to the right as the conclusion, even though strictly speaking this is not necessary. Let's highlight the premises and the rows where they are both true:

p	q	$\neg p$	$\neg p \vee q$	q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

Now we check the conclusion's truth value in that row:

p	q	$\neg p$	$\neg p \vee q$	q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

So this argument is valid: because whenever the premises are both true, so is the conclusion.

Exercise 4.4 Use truth tables to check the validity of the following arguments.

$p \vee q$	$p \wedge \neg q$	$\neg q \vee \neg p$	$p \wedge q$
$\neg p$	$\neg\neg q$	$p \wedge r$	$q \vee r$
$\therefore q$	$\therefore r$	$\therefore \neg q$	$\therefore p \wedge (q \vee r)$

Do you notice anything odd about the second argument? ■

4.4 Tautologies

Definition 4.4.1 — Tautology. In propositional logic, a tautology is any sentence that has only T as truth value.

■ **Example 4.3** Consider sentence $p \vee \neg p$ and construct its truth table:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

There is just *no way* $p \vee \neg p$ can be false. In other words, the compound sentence $p \vee \neg p$ is true *regardless* of the truth value of its components. It is a tautology. ■

■ **Example 4.4** Here is another example: $q \vee (p \vee \neg p)$. Fill in its truth table:

p	q	$\neg p$	$p \vee \neg p$	$q \vee (p \vee \neg p)$
T	T	F	T	...
T	F	F	T	...
F	T	T	T	...
F	F	T	T	...

Is there any row in which $q \vee (p \vee \neg p)$ is false? So is the sentence a tautology? ■

R In propositional logic, it is only compound sentences that can be tautologies. Why might that be?

4.5 Homework

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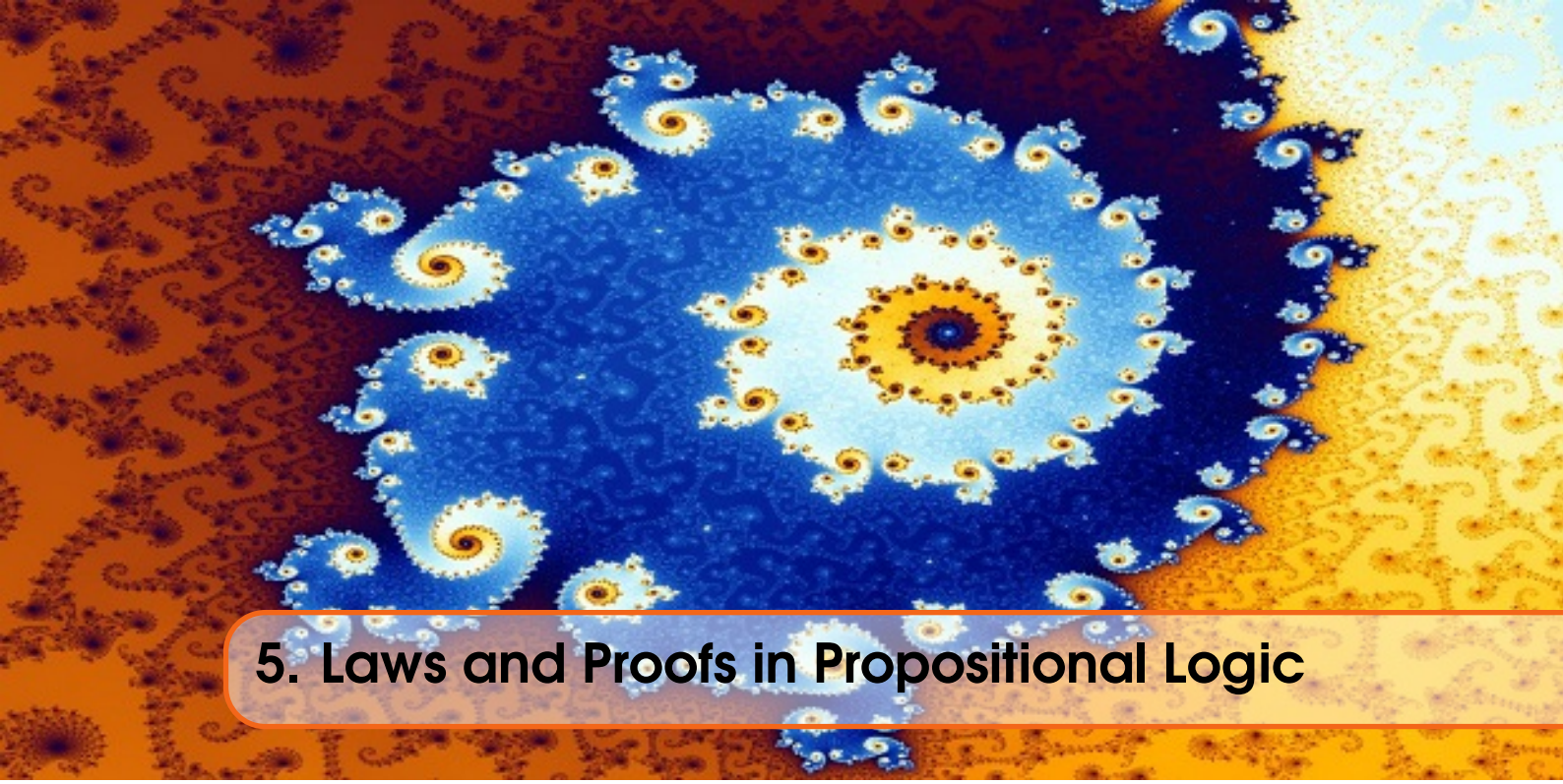
4.6 Notes



5. Laws and Proofs in Propositional Logic

5.1 Introduction

In this chapter we will look at the laws of propositional logic, as well as to how to go about proving them informally (recall the distinction between formal and informal proof sketched at the beginning of Chapter 3).

LEARNING OBJECTIVES

At the end of this chapter you will:

- be familiar with symbols $\top$, $\bot$, $\leftrightarrow$;
- be familiar with the laws of propositional logic;
- be satisfied that they are tautologies;
- be able to read and understand informal proofs in propositional logic;
- have tried to work out simple informal proofs in propositional logic.

5.2 Three more Symbols

We will now look at the laws of propositional logic. As you will appreciate, they very closely resemble the laws of the algebra of sets. This has a profound mathematical meaning; I will not go into that here, although we might talk about it in the course.

Before we do so, however, three new symbols must be introduced.

Definition 5.2.1 — Tautology symbol. $\top$ is the tautology symbol. It stands for any tautology.

R $\top$ stands for *any* tautology because all tautologies have the same truth value, namely, T. For this reason, $\top$ is sometimes called the 'truth symbol' – though that is somewhat imprecise.

Definition 5.2.2 — Contradiction symbol. $\bot$ is the contradiction symbol. It stands for any contradiction.

R $\perp$ stands for *any* contradiction because all contradictions have the same truth value, namely, F. For this reason, $\perp$ is sometimes called the 'falsehood symbol' – though that is somewhat imprecise.

R $\top$ and $\perp$ are often regarded as zero-place connectives. This is stretching the notion of connective extremely thin – but you see a lot of this in mathematics. For example, a point is just a circle with radius 0.

The third symbol is, strictly, speaking, a logical connective; therefore, it should be defined via its truth table. However, I will also give a discursive definition. The latter should be enough for our purposes; I will give both for the sake of completeness.

Definition 5.2.3 — Equivalence, $\leftrightarrow$. Equivalence, written $\leftrightarrow$, is a two-place logical connective. For any two sentences p, q, the compound sentence $p \leftrightarrow q$ is true if and only if p is true whenever q is true. It has the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

R Notice that $\leftrightarrow$ has a similar truth table as $\wedge$. There is, however, an important difference. When both p and q are false (last row), $p \wedge q$ is false (check the relevant definition in Chapter 3), while $p \leftrightarrow q$ is true. This reflects the underlying idea of logical equivalence: $p \leftrightarrow q$ is true when p and q have always the same truth values.

R We can now say something more precise about $\top$ and $\perp$. $\top$ is *equivalent* to every tautology, and $\perp$ is *equivalent* to every contradiction.

5.3 The Laws of Propositional Logic

Laws of Propositional Logic — Associative laws

L1.1 $((p \wedge q) \wedge r) \leftrightarrow (p \wedge (q \wedge r))$

L1.2 $((p \vee q) \vee r) \leftrightarrow (p \vee (q \vee r))$

Laws of Propositional Logic — Commutative laws

L2.1 $(p \wedge q) \leftrightarrow (q \wedge p)$

L2.2 $(p \vee q) \leftrightarrow (q \vee p)$

Laws of Propositional Logic — Absorption Laws

L3.1 $(p \wedge (p \vee q)) \leftrightarrow p$

L3.2 $(p \vee (p \wedge q)) \leftrightarrow p$

Laws of Propositional Logic — Distributive Laws

L4.1 $(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$

L4.2 $(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$

Laws of Propositional Logic — Neutral Elements Laws

L5.1 $(p \wedge \top) \leftrightarrow p$ **L5.2** $(p \wedge \bot) \leftrightarrow \bot$

L5.3 $(p \vee \top) \leftrightarrow \top$ **L5.4** $(p \vee \bot) \leftrightarrow p$

Laws of Propositional Logic — Negation Laws

L6.1 $(p \wedge \neg p) \leftrightarrow \bot$

L6.2 $(p \vee \neg p) \leftrightarrow \top$

R As you can see, $\wedge$ has replaced $\cap$, $\vee$ has replaced $\cup$, $\neg$ has replaced $'$, $\top$ has replaced U, $\bot$ has replaced $\emptyset$, and $\leftrightarrow$ has replaced $=$.

Watch out, though: although the negation of $=$ is $\neq$, there is no negation of $\leftrightarrow$ as such. In order to write 'p and q are not equivalent', you need to write $\neg(p \leftrightarrow q)$.

Just as with the laws of the algebra of sets, you need to familiarise yourself with the laws of propositional logic. You should try and read them while always translating them into English. Take, for example, the first Neutral Elements Law, L5.1: $(p \wedge \top) \leftrightarrow p$. In English, it means that the conjunction of any sentence p and any tautology whatsoever is equivalent to p (i.e., it is true whenever p is).

Exercise 5.1 Choose two Laws of Propositional Logic and translate them into English. ■

5.4 The Laws of Propositional Logic as Tautologies

The laws are such because they are true regardless of the truth value of their component sentences – because they are tautologies. You know from last chapter how to check tautologies by using truth tables. Let's try to check that L5.1 is.

Before doing so, notice that the truth tables of $\top$ and $\bot$ are made up, respectively, of T and F only. (Why is that so?)

Also notice that you have to ways of going about checking the truth table of a sentence such as $(p \wedge \top) \leftrightarrow p$. You can either construct the sentence's truth table, based on Definition 5.2.3; or work on the left-hand and the right-hand sides – $p \wedge \top$ and p – independently: construct the two's

truth tables then compare them. We will take the latter course here, but you are welcome to take the former.

The truth table of $p \wedge \top$ is as follows:

p	$\top$	$p \wedge \top$
T	T	T
T	T	T
F	T	F
F	T	F

The truth table of p is simply the first column of the above table. If we highlight the relevant columns:

p	$\top$	$p \wedge \top$
T	T	T
T	T	T
F	T	F
F	T	F

we can see quite clearly that they are identical. Therefore, the two sentences – p and $p \wedge \top$ – will be true and false in exactly the same circumstances. Therefore, $(p \wedge \top) \leftrightarrow p$ will always be true: because it is true whenever the right-hand and the left-hand sides are either both true or both false, and thus whenever they have the same truth table. We can check that that is so by actually constructing the truth table of $(p \wedge \top) \leftrightarrow p$:

p	$\top$	$p \wedge \top$	$(p \wedge \top) \leftrightarrow p$
T	T	T	T
T	T	T	T
F	T	F	T
F	T	F	T

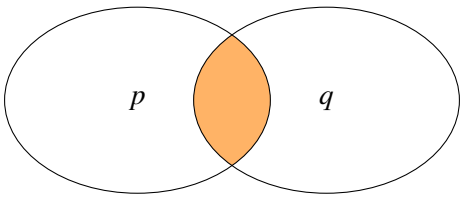
So law L5.1 is in fact a tautology.

Exercise 5.2 Check that all the Laws of Propositional Logic are tautologies. ■

5.5 How to prove Sentences in Propositional Logic

Proofs in propositional logic are very similar to proofs in the algebra of sets (at least informal proofs – recall the distinction I sketched at the beginning of Chapter 3), so it will probably not take much for you to work your head around them.

Before we start, however, notice that you can take advantage of the similarity between the logical connectives and the set-theoretical operations and visualise sentences through Venn diagrams. For example, $p \wedge q$ may be represented as an intersection:



– the idea being that each ellipse is the space where p and q, respectively, are true, while the shaded area is the space where *both* p and q are true, i.e., their conjunction.

Exercise 5.3 How would you represent disjunction and negation in terms of Venn diagrams? Sketch your diagrams here. ■

Although representation in terms of Venn diagrams can give you some insight into the Laws of Propositional Logic, it is no proof. Informal proofs in propositional logic work just like proofs in the algebra of sets. Typically, you prove the left-to-right and the right-to-left directions separately (the clue is in the equivalence symbol) – just as we did with sets. Also, you do so by always going back to the definition of the connectives and the $\top$ and $\bot$ symbols.

Here is an example:

Proof L3.1: $(\mathbf{p} \wedge (\mathbf{p} \vee \mathbf{q})) \leftrightarrow \mathbf{p}$

Prove that $(p \wedge (p \vee q)) \leftrightarrow p$.

Proof:

Left to right:

1. Assume that $p \wedge (p \vee q)$ (Assumption);
2. then both p and $p \vee q$ (by Definition of $\wedge$);
3. then p (by step 2). □

Right to left:

4. Assume that p (assumption);
7. then $p \vee q$ (by Def. $\vee$);
6. if both p and $p \vee q$, then $p \wedge (p \vee q)$ (by Def. $\wedge$). ■

Here is another example:

Proof L6.1: $(\mathbf{p} \wedge \neg \mathbf{p}) \leftrightarrow \bot$

Prove that $(p \wedge \neg p) \leftrightarrow \bot$.

Proof:

1. Assume that $(p \wedge \neg p)$ (Assumption);
2. then both p and $\neg p$ (Def $\wedge$);
3. if p, then $\neg p$ is false, and if $\neg p$ then p is false (Def $\neg$);
4. p and $\neg p$ can't both be true (clarification);
5. Therefore, $\bot$ (2,3) ■

Notice that in this case it was unnecessary to distinguish between two directions: we exhausted all cases in steps 1-3 and then derived a contradiction – and by definition, $\bot$ just is equivalent to any contradiction.

If we wanted to, however, we could derive $p \wedge \neg p$ from $\bot$. Indeed, we could have derived any sentence whatsoever. This is because of a rule of inference in classical propositional logic known as *ex falsum quodlibet*, which is Latin for 'anything follows from a falsehood (contradiction)'. It means that if, in a proof, you reach a contradiction at step n, at step $n+1$ you can put any sentence whatsoever and it will be a correct inference.

Exercise 5.4 Think about the definition of validity given in Chapter 1. Do you think there is any connection between that and the *ex falsum quodlibet* rule? ■

Proof by contradiction is also available in propositional logic. Recall that a law such as $(p \wedge \neg p) \leftrightarrow \bot$ simply means that the conjunction of a sentence and its negation is always false (it is a contradiction). Its dual law, L6.2: $(p \vee \neg p) \leftrightarrow \top$, then, means that the disjunction of a sentence and its negation is a tautology. One way to prove L6.2, then, is to assume the negation of the left-hand side of the law, namely $p \vee \neg p$, derive a contradiction, and conclude that since the negation of $p \vee \neg p$ is a contradiction, then $p \vee \neg p$ must be a tautology. And since L6.2 precisely says that $p \vee \neg p$ is a tautology, you will have proven the law itself.

Proof L6.2: $(\mathbf{p} \vee \neg \mathbf{p}) \leftrightarrow \top$

Prove that $(p \vee \neg p) \leftrightarrow \top$.

Proof:

1. Assume that $\neg(p \vee \neg p)$ (Assumption);
2. then neither p nor $\neg p$ (Defs $\vee$, $\neg$);
3. if p is false, then $\neg p$ is true; and if $\neg p$ is false, then p is true (Def $\neg$);
4. p and $\neg p$ can't both be false (clarification);
5. $\bot$ (2,3)
6. then $\neg\neg(p \vee \neg p)$ (i.e., it is false that $\neg(p \vee \neg p)$); (1,5)
7. therefore, (it is true that) $p \vee \neg p$. ■

Exercise 5.5 Choose three Laws of Propositional Logic (except those we have already proven) and have a go at proving them. ■

5.6 Homework

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5.7 Notes

6. More Proofs and Final Assignment

6.1 Introduction

This chapter is all about practising more proofs. I will give you six tautologies and ask you to prove as many as you can. One of these will be an example proof. Make sure you translate every tautology into English before you prove it. Have a go at experimenting with different proof strategies (direct proof and proof by contradiction).

Keep in mind that this chapter's homework is your final assignment.

LEARNING OBJECTIVES
At the end of this chapter you will:
- have practised more proofs of tautologies.

6.2 Proof 1 (Example)

Proof $p \leftrightarrow (p \wedge p)$

Prove that $p \leftrightarrow (p \wedge p)$.
In English: p is equivalent to the conjunction of p and itself.
Proof:

Left to right:

1. Assume that p (Assumption);
2. then $p \wedge p$ (1, Def. $\wedge$);
3. if p is true, then the conjunction of p and itself must also be true (clarif.); $\square$

Right to left:

4. assume that $p \wedge p$ (assumption);
5. then p (4, Def. $\wedge$)
6. if $p \wedge p$ is true, then by definition both conjuncts are true; so p must be true (clarification). $\blacksquare$

6.3 Proof 2

Proof $\neg(p \wedge q) \vee p$

6.4 Proof 3

Proof $\neg p \vee (p \vee q)$

6.5 Proof 4

Proof $p \leftrightarrow (p \vee p)$

6.6 Proof 5

Proof $p \leftrightarrow \neg\neg p$

6.7 Proof 6

Proof $(\neg p \vee q) \leftrightarrow (q \vee \neg p)$

6.8 Homework

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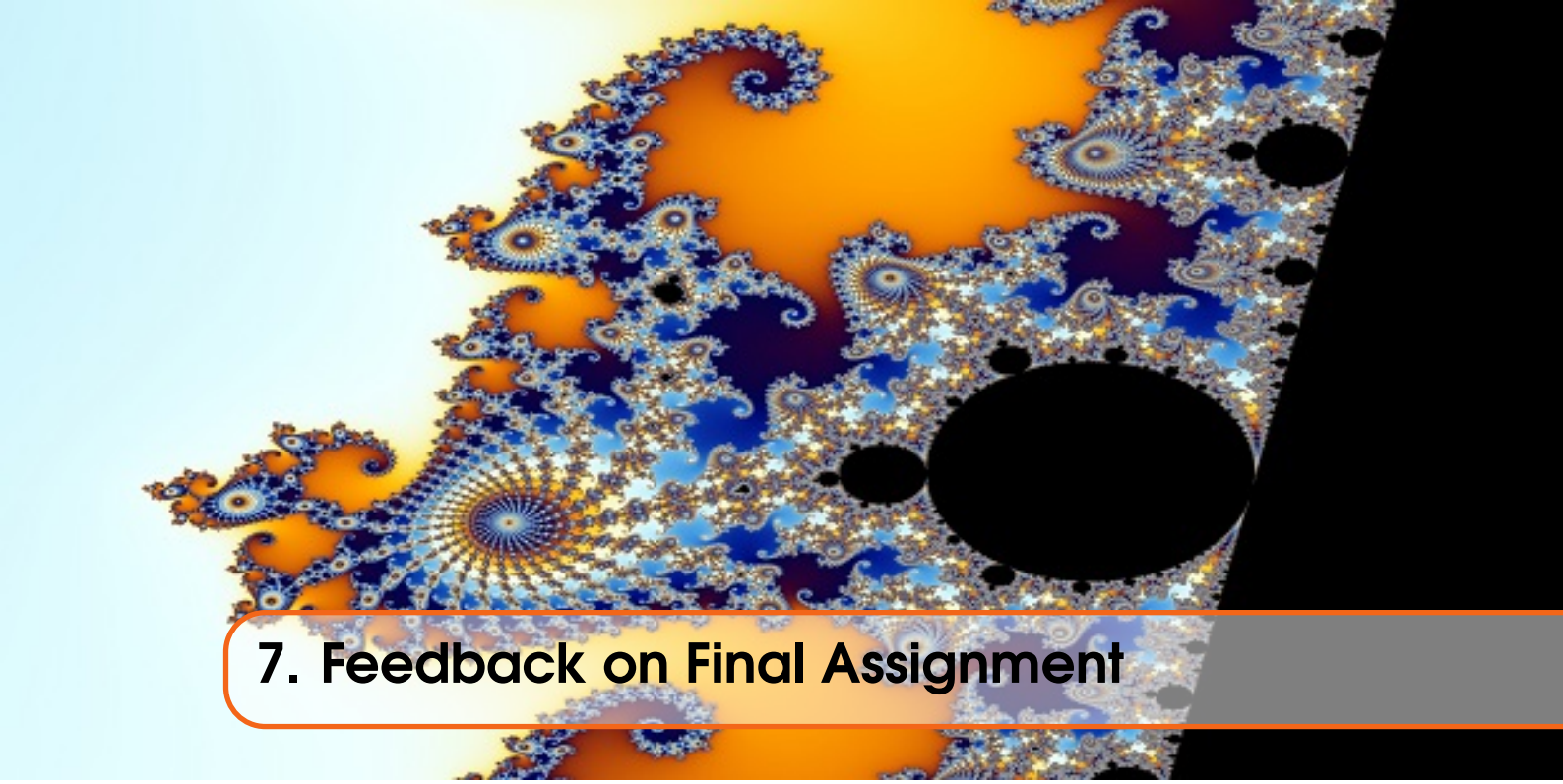
6.9 Notes



7. Feedback on Final Assignment

[This is where you should attach your feedback once you have received and annotated it. You then have two pages for additional notes.]

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